

Weight structures vs. t -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)

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Abstract

In this paper we introduce a new notion of a *weight structure* (w) for a triangulated category \underline{C} ; this notion is an important natural counterpart to the notion of a t -structure. This allows to extend several results of the preceding paper [Bon09] to a large class of triangulated categories and functors.

The *heart* of w is an additive $\underline{Hw} \subset \underline{C}$. We prove that a weight structure defines Postnikov towers for any $X \in \text{Obj } \underline{C}$ (whose 'factors' are $X^i \in \text{Obj } \underline{Hw}$). For any (co)homological functor $H : \underline{C} \rightarrow A$ (A is abelian) these towers yield *weight spectral sequences* $T : H(X^i[j]) \implies H(X[i+j])$; T is canonic and functorial starting from E_2 . T specializes to the usual (Deligne's) weight spectral sequences for 'classical' realizations of Voevodsky's motives DM_{gm}^{eff} (if we consider $w = w_{Chow}$ with $\underline{Hw} = Chow^{eff}$) and to Atiyah-Hirzebruch spectral sequences in topology.

We prove that there often exists an exact conservative weight complex functor $\underline{C} \rightarrow K(\underline{Hw})$. This is a generalization of the functor $t : DM_{gm}^{eff} \rightarrow K^b(Chow^{eff})$ constructed in [Bon09] (which is an extension of the weight complex of Gillet and Soulé). We prove that $K_0(\underline{C}) \cong K_0(\underline{Hw})$ under certain restrictions.

We also introduce the concept of adjacent structures: a t -structure is adjacent to w if their negative parts coincide. This is the case for the Postnikov t -structure for the stable homotopy category SH (of topological spectra) and a certain weight structure for it that corresponds

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to cellular filtration. We also define a new (*Chow*) t -structure t_{Chow} for $DM_{gm}^{eff} \supset DM_{gm}^{eff}$ which is adjacent to the Chow weight structure mentioned. We have $\underline{Ht}_{Chow} \cong \text{AddFun}(Chow^{eff op}, Ab)$; t_{Chow} is related to unramified cohomology. Functors left adjoint to those that are t -exact with respect to some t -structures are weight-exact with respect to the corresponding adjacent weight structures, and vice versa. Adjacent structures identify two spectral sequences converging to $\underline{C}(X, Y)$: the one that comes from weight truncations of X with the one coming from t -truncations of Y (for $X, Y \in \text{Obj}\underline{C}$). Moreover, the philosophy of adjacent structures allows to express torsion motivic cohomology of certain motives in terms of étale cohomology of their 'submotives'. This is an extension of the calculation of E_2 of coniveau spectral sequences (by Bloch and Ogus).

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Introduction

The purpose of this paper is twofold: (i) the introduction of a new formalism of *weight structures* for triangulated categories; (ii) the application of the general theory obtained to Voevodsky's motives (we also consider the stable homotopy category SH of topological spectra).

The notion of a weight structure w for a triangulated \underline{C} is an important natural counterpart to the notion of a t -structure; also, these two types of structures are connected by interesting relations. Similarly to t -structures, weight structures are described in terms of $\underline{C}^{w \leq 0}$ (the non-positive part of \underline{C}) and $\underline{C}^{w \geq 0}$ (the non-negative part). We demand: any object can be 'decomposed' (non-uniquely) into a non-positive and a non-negative part; we call the corresponding distinguished triangles *weight decompositions*. The heart \underline{Hw} of w is defined as the intersection of these two parts.

The most simple (yet quite interesting) example of a weight structure is: $\underline{C} = K^?(B)$ (the homotopy category of complexes over an additive B ; ? is any boundedness condition). We define $\underline{C}^{w \leq 0}$ (resp. $\underline{C}^{w \geq 0}$) as the class of complexes that are homotopy equivalent to those concentrated in non-positive (resp. non-negative) degrees. Then \underline{Hw} contains B (it is equivalent to B if the latter is pseudo-abelian). Weight decompositions are given by stupid truncations of complexes (so any object of \underline{C} has a large class of non-equivalent weight decompositions). In this case our theory shows what functorial information could be obtained using stupid truncation of complexes (this is a significant amount of information that was never considered in the literature). Note that this example of a weight structure is 'almost universal' if one fixes the heart: for a 'reasonable' \underline{C} endowed with a weight structure one has an exact *weight complex* functor $\underline{C} \rightarrow K(\underline{Hw})$ (these comparison functors exist much more often than their t -structure counterparts i.e. functors $D^b(\underline{Ht}) \rightarrow \underline{C}$ for \underline{Ht} being the heart of a t -structure for \underline{C}); the weight complex is 'usually' conservative (though usually not an isomorphism).

In this example (as well as in the general case) there are no \underline{C} -morphisms of positive degrees (one may also call them positive Ext's with respect to \underline{C}) between objects of \underline{Hw} (note in contrast that there are no morphisms of

negative degrees between objects of the heart of a t -structure); we will call such subcategories of \underline{C} *negative*. In fact, if w is *bounded* (as in the case of $K^b(B)$) then it can be easily recovered from \underline{Hw} . Moreover, any negative additive $B \subset \underline{C}$ can be uniquely extended to a weight structure (at least) in the case when \underline{C} is generated by B as a triangulated category.

This observation allows to construct weight structures in two following important (and more complicated) cases. The first case is $\underline{C} = DM_{gm}^{eff}$ (the Voevodsky's category of effective geometric motives), $\underline{Hw} = Chow^{eff}$ (effective Chow motives); we call the corresponding weight structure w_{Chow} the *Chow weight structure*. In this case our theory yields canonical DM_{gm}^{eff} -functorial (Chow)-weight spectral sequences $T(H, X)$ for any cohomological functor $H : DM_{gm}^{eff} \rightarrow A$ (and $X \in Obj DM_{gm}^{eff}$). T relates cohomology of Voevodsky's motives with those of Chow motives; it is a vast (and 'motivic') generalization of the usual (i.e. Deligne's) weight spectral sequences (for étale and singular cohomology of varieties). In particular, we obtain certain (Chow)-weight spectral sequences for motivic cohomology. We also have (as was said above) an exact conservative weight complex functor $t : DM_{gm}^{eff} \rightarrow K^b(Chow^{eff})$ (it extends to a functor $DM_{gm} \rightarrow K^b(Chow)$; the restriction of t to $Chow^{eff} \subset DM_{gm}^{eff}$ is the obvious embedding $Chow^{eff} \rightarrow K^b(Chow^{eff})$).

Note that it is traditionally expected that the weight filtration for singular and étale cohomology of motives is induced by a certain *weight filtration* of DM_{gm}^{eff} (or $DM_{gm}^{eff}\mathbb{Q}$). We briefly explain the relation of this (conjectural) picture with our results. It is conjectured that singular and étale cohomology for DM_{gm}^{eff} can be factorized through the category MM (of mixed motives) which is the heart of a certain mixed motivic t -structure for DM_{gm}^{eff} . It is easily seen that the restriction of w_{Chow} to MM 'should yield' the restriction of the weight filtration to MM . When the base field is a number field, J. Wildeshaus justified this picture for the restriction of w_{Chow} to *Artin-Tate motives*; see [Wil09at] and §8.6. One can also describe a non-conjectural (and compatible) analogue of this picture for mixed Hodge structures (lying inside the triangulated category of Hodge complexes).

The second case is: $\underline{C} = SH_{fin}$, \underline{Hw} consists of finite sums of the sphere spectrum. In this case weight decompositions are induced by cellular filtration; weight spectral sequences yield Atiyah-Hirzebruch spectral sequences!

Now we describe some more constructions and results (in the literature) that are closely related with weight structures.

The simplest and oldest of them are: projective and injective (hyper)resolutions of objects of (or complexes over) an abelian category A and the description of derived functors; note here that $D^?(A)$ is 'often' isomorphic to $K^?(Proj A)$ (and to $K^?(Inj A)$). This example is related to the stupid weight structure of the latter category. Note that stupid truncation of complexes does not

yield a functor on $K(\text{Proj } A)$; still in terms of it one can describe derived functors.

Another example is: Deligne's weight filtrations and spectral sequences (for étale and singular cohomology of varieties). Note that weight spectral sequences in [Del74] depended on the choice of nice compactifications of smooth quasi-projective varieties and proper smooth hypercoverings of non-smooth projective varieties; Deligne's theory of weights yields the functoriality of the result only after tensoring by \mathbb{Q} . In [GiS96] it was proved that the E_2 -terms of spectral sequences mentioned factorize through the weight complex functor (this term was also introduced by Gillet and Soulé); this allowed to define functorial weight spectral sequences with integral coefficients. Finally, in the preceding paper [Bon09] certain weight spectral sequences were defined (and described completely) for a wide class of cohomological functors defined on DM_{gm}^{eff} . Note here that the results of [Bon09] were deduced from the fact that DM_{gm}^{eff} has a certain (negative) differential graded description (in terms of *twisted complexes* introduced in [BoK90]); they can be easily extended to any category that has such a description. The description of DM_{gm}^{eff} mentioned is somewhat similar to the definition of Hanamura's motives (see [Han99]; in [Bon09] it was also proved that Hanamura's motives are anti-isomorphic to Voevodsky's ones); so it is no wonder that some of the constructions of [Bon09] (including certain stupid truncations for motives) are similar to those of [Han99]. In the current paper we prove that weights can be introduced for any (co)homological functor $DM_{gm}^{eff} \rightarrow A$ (A is an abelian category).

(More obvious) predecessors of the definition of a weight structure were the classical notions of 'filtration bette' (see §3.1.7 of [BBD82]) and of connective spectra (see §7 of [HPS97]). Possibly, the so-called cofiltrations considered in [B-VRS03] are also related to the subject. Still, our axiomatics and most of main results are completely new. The only exception known to the author is that a part of Theorem 4.5.2 (the one that concerns t -structures) is a slight generalization of Theorem 1.3 of [HKM02]. Besides, recently weight structures were also (independently) introduced by D. Pauksztello (he called them co- t -structures; see [Pau08]); some of (easier) results of the current paper were also proved there. We would also like to mention the paper [Wil09c] where our results are applied to the study of so-called boundary motives.

Half of the results of the current paper are extensions of the results of [Bon09] to the case of arbitrary triangulated categories endowed with weight structures (that do not necessarily have a differential graded description). We define weight spectral sequences and complexes. In the bounded case we prove that \underline{C} is pseudo-abelian whenever \underline{Hw} is; in this case also $K_0(\underline{C}) \cong K_0(\underline{Hw})$. We also define and study certain $K_0(\text{End } \underline{C})$; this allows to calculate $K_0(\text{End } SH_{fin})$ very explicitly.

The other half of the results of the current paper is related to the yoga of *adjacent structures*. This concept seems to be completely new (though some examples for it are well-known and very important, it seems that nobody studied this aspect of them). We will say that w is *left adjacent* to a t -structure t if $\underline{C}^{w \leq 0} = \underline{C}^{t \leq 0}$. The most simple (yet non-trivial) example here is the canonical t -structure for $\underline{C} = D^2(A)$ and the (stupid) weight structure coming from the isomorphism $\underline{C} \cong K^?(Proj A)$ (if there is one). Another example is the *spherical* weight structure for SH (mentioned above) and the Postnikov t -structure for it; in this case our formalism implies that both of these structures yields the same Atiyah-Hirzebruch spectral sequence converging to $SH(X, Y)$ ($X, Y \in Obj SH$). Also, we extend w_{Chow} from DM_{gm}^{eff} to DM_-^{eff} and construct t_{Chow} that is adjacent to it. t_{Chow} is related to unramified cohomology; $Ht_{Chow} \cong AddFun(Chow^{eff op}, Ab)$. An illustration of relevance of this notion: functors left adjoint to those that are t -exact with respect to some t -structures are *weight-exact* with respect to the corresponding adjacent weight structures, and vice versa.

We relate t -truncations and weight truncations for adjacent structures; this gives a collection of new interesting formulae. As in the partial case $\underline{C} = SH$, we also have: for $X, Y \in Obj \underline{C}$ the spectral sequence whose E_2 -terms are given by morphisms groups from X to the t -cohomology of Y , is isomorphic to the weight spectral sequence $T(\underline{C}(-, Y))$ (applied to X); see Theorem 2.6.1 of [Bon10]. The corresponding exact couples are isomorphic also. These spectral sequence calculations are closely related to the well-known calculations of the E_2 -terms of coniveau spectral sequences (by Bloch and Ogus in [BOg94]: see also [C-THK]); we extend the latter and generalize them to certain motives. This allows to express torsion motivic cohomology of these motives in terms of étale cohomology of their 'submotives' (in a certain sense).

In the next paper [Bon10] we also construct a certain *Gersten weight structure* for a certain category of *comotives* $\mathfrak{D}_s \supset DM_{gm}^{eff}$ (under a technical restriction that k is countable; see part 4 of Remark 7.4.3 and Remark 7.1.2 below). This result allows to extend the coniveau results mentioned to arbitrary motives and makes them more functorial; yet for general motives it seems difficult to calculate the corresponding ('Gersten') weight decompositions explicitly.

We also develop the (general) theory of weight structures further in §2 of [Bon10]. There we study *orthogonal* weight and t -structures with respect to a (*nice*) *duality* $\underline{C}^{op} \times \underline{D} \rightarrow A$; this is a generalization of the notion of adjacent structures. We relate weight spectral sequences and virtual t -truncations of functors (introduced in the current paper) with t -structures in this more

general situation (see also Remark 4.4.3 below).

Now we list the contents of the paper. More details could be found at the beginnings of sections.

In section 1 we give the definition of a weight structure. We also give some more basic definitions and prove their (relatively) simple properties. Our central objects of study are *weight decompositions* of objects and morphisms. We also describe certain (*weight*) Postnikov towers for objects of \underline{C} that come from weight structures.

In section 2 we define the *weight spectral sequence* $T(H, X)$ (for $X \in \text{Obj}\underline{C}$ and a (co)homological functor $H : \underline{C} \rightarrow A$); it comes from a (weight) Postnikov tower mentioned. T is canonic and functorial starting from E_2 . It specializes to the 'usual' weight spectral sequences for 'classical realizations' of Voevodsky's motives (at least with rational coefficients). Moreover, in this case T degenerates at E_2 , and $E_2^{*,*}$ are exactly the graded pieces of the weight filtration of $H^*(X)$.

We also study the D -terms of the derived exact couple for $T(F, X)$ (for a (co)homological F). For a homological F they equal $F_2(X) = \text{Im}(F(w_{\geq k+1}X) \rightarrow F(w_{\geq k}X))$ (or $F_1(X) = \text{Im}(F(w_{\leq k+1}X) \rightarrow F(w_{\leq k}X))$ for the other possible version of the exact couple; see Remark 2.3.3) in our notation. F_1 and F_2 are both (co)homological; they behave as if they are given by truncations of F in some triangulated 'category of functors' \underline{D} with respect to some t -structure. Composing these *virtual t -truncations* 'from different sides' one obtains $E_2^{*,*}(T)$.

In section 3 we define the weight complex functor t . Its target is a certain *weak category of complexes* $K_{\mathfrak{w}}(\underline{Hw})$. $K_{\mathfrak{w}}(\underline{Hw})$ is a factor of $K(\underline{Hw})$ which is no longer triangulated. Yet the kernel of the projection $K(\underline{Hw}) \rightarrow K_{\mathfrak{w}}(\underline{Hw})$ is an ideal of morphisms whose square is zero so our (weak) weight complex functor is not much worse than the 'strong' one (as constructed in [Bon09] in the differential graded case). In particular, t is conservative, *weakly exact*, and preserves the filtration given by the weight structure (in the bounded case). We conjecture that the strong weight complex functor exists also; see Remark 3.3.4 and §8.4. Besides, in some cases (for example, for $SH_{fin} \subset SH$) we have $K_{\mathfrak{w}}(\underline{Hw}) = K(\underline{Hw})$. Our main tool of study is the *weight decomposition functor* $WD : \underline{C} \rightarrow K_{\mathfrak{w}}^{[0,1]}(\underline{C})$; see Theorem 3.2.2.

A reader only interested in motives could skip this section since we construct the strong version of the weight complex for motivic categories in §6.

In section 4 we relate weight structures with t -structures (via the notion of *adjacent structures*), construct and study examples of this notion. In the case when weight and t -structures are *adjacent*, we have a certain duality of their hearts, whereas spectral sequences coming from these structures are

closely related (they differ only by a certain shift of indices). A functor left adjoint to a t -exact one is *weight-exact* (if we consider left adjacent weight and t -structures). We also prove that a weight structure can often be described in terms of some *negative* additive subcategory of \underline{C} .

In §4.6 we apply our results to the study of the stable homotopy category. The corresponding *spherical* weight structure constructed is generated by the sphere spectrum; it is *left adjacent* to the (usual) Postnikov t -structure on SH . Postnikov towers corresponding to this weight structure are called *cellular towers* (by topologists). It turns out that the corresponding weight complex calculates the singular (co)homology of spectra in this case.

In section 5 we prove that a bounded \underline{C} is idempotent complete whenever \underline{Hw} is; the idempotent completion of a general bounded \underline{C} has a weight structure whose heart is the idempotent completion of \underline{Hw} . If \underline{C} is bounded and idempotent complete then $K_0(\underline{C}) \cong K_0(\underline{Hw})$. In §5.4 we study a certain Grothendieck group of endomorphisms in \underline{C} . Though it is not always isomorphic to $K_0(\text{End } \underline{Hw})$, it is if \underline{Hw} is *regular* in a certain sense. Besides, we can still say something about $K_0(\text{End } \underline{C})$ in the general case also. In particular, this allows us to generalize Theorem 3.3 of [BLE07] (on independence of l for traces of certain *open correspondences*); see also §8.4 of [Bon09]. As an application of our results, we also calculate explicitly the groups $K_0(SH_{fin})$ and $K_0(\text{End } SH_{fin})$ (along with their ring structure). We also extend these results to the calculation of certain $K_0(\text{End}^n SH_{fin})$ for $n \in \mathbb{N}$.

In section 6 we translate the results of [Bon09] into the language of weight structures. In particular, we show that Voevodsky's $DM_{gm}^{eff} (\subset DM_{gm})$ admits a *Chow weight structure* whose heart is $Chow^{eff}$ (resp. $Chow$). This allows us to prove that (Chow)-weight spectral sequences for realizations (almost the same as those constructed in §7 of [Bon09]; see §6.4) exist for all realizations and do not depend on any choices.

In section 7 we show that the Chow weight structure of DM_{gm}^{eff} extends to DM_-^{eff} and admits an adjacent t -structure t_{Chow} (whose heart is the category $Chow_*^{eff} = \text{AddFun}(Chow^{eff, op}, Ab) \supset Chow^{eff}$). t_{Chow} is closely related to unramified cohomology! We also prove that any possible (conjectural) mixed motivic t -structure for $DM_{gm}^{eff} \mathbb{Q}$ would automatically induce a canonical weight filtration on its heart (i.e. on mixed motives). We also prove that (a certain version of) the weight complex functor can be defined on $DM_{gm}^{eff} \subset DM_-^{eff}$ without using the resolution of singularities (so one can define it for motives over any perfect field).

Next, we apply the philosophy of adjacent structures to the study of coniveau spectral sequences. We express the cohomology of a motif X with coefficients in the homotopy (t -structure) truncations of an arbitrary $H \in$

$ObjDM_-^{eff}$ in terms of the limit of H -cohomology of certain 'submotives' of X . These calculations are closely related with the well-known results of Bloch and Ogus on E_2 of the coniveau spectral sequence; see [BOg94], [C-THK], and also [Par96]. In particular, one can express torsion motivic cohomology of certain motives in terms of étale cohomology of their 'submotives' (this requires Beilinson-Lichtenbaum conjecture). As a partial case, we obtain a formula for the (torsion) motivic cohomology with compact support of a smooth quasi-projective variety.

In section 8 we show that a weight structure w on \underline{C} which induces a weight structure on a triangulated $\underline{D} \subset \underline{C}$ yields also a weight structure on the Verdier quotient $\underline{C}/\underline{D}$. We also prove that weight structures can be glued in a manner that is similar to those for t -structures. This fact would possibly be used for the construction of the theory of weights for relative motives (Voevodsky's motives over a base that is not a field) in a forthcoming paper.

Next we observe a funny thing: functors represented by compositions of t -truncations with respect to distinct t -structures can be expressed in terms of the corresponding adjacent weight structures (as certain images). We prove (by an argument due to A. Beilinson) that any f -category enhancement of \underline{C} yields a 'strong' weight complex functor $\underline{C} \rightarrow K(\underline{Hw})$. We also describe other possible sources of conservative 'weight complex-like' functors (they are usually conservative) and related spectral sequences. We conclude by the discussion on relevant types of filtrations for triangulated categories, and of the conjectural picture of this for $DM_{gm}^{eff}\mathbb{Q}$.

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Notation. For a category C , $A, B \in ObjC$, we denote by $C(A, B)$ the set of C -morphisms from A to B .

For categories C, D we write $C \subset D$ if C is a full strict subcategory of D . Recall that D is called *strict* if it contains all objects in $ObjC$ isomorphic to those from $ObjD$.

For a category C , $X, Y \in ObjC$, we say that X is a *retract* of Y if id_X can be factorized through Y . Note: if C is triangulated or abelian then X is a retract of Y whenever X is its direct summand. For an additive $D \subset C$ the subcategory D is called *Karoubi-closed* in C if it contains all retracts of its objects in C .

$X \in ObjC$ will be called compact if the functor $X^* = C(X, -)$ commutes with all those small coproducts that exist in C (contrary to the tradition, we do not assume that arbitrary coproducts exist).

For a category C we denote by C^{op} the opposite category.

\underline{C} will usually denote a triangulated category; usually it will be endowed with a weight structure w (see Definition 1.1.1 below). We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a functor that preserves the structures of triangulated categories). We will call a covariant additive functor $\underline{C} \rightarrow A$ for an abelian A *homological* if it converts distinguished triangles into long exact sequences; homological functors $\underline{C}^{op} \rightarrow A$ will be called *cohomological* when considered as contravariant functors $\underline{C} \rightarrow A$.

For $f \in \underline{C}(X, Y)$, $X, Y \in \text{Obj}\underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \rightarrow Z$ a cone of f . Recall that different choices of cones are connected by non-unique isomorphisms (easy, see IV.1.7 of [GeM03]). Besides, in $C(A)$ (see below) we have canonical cones of morphisms, (see section §III.3 of *ibid.*).

We will often specify a distinguished triangle by two of its morphisms.

When dealing with triangulated categories we (mostly) use conventions and auxiliary statements of [GeM03]. For a set of objects $C_i \in \text{Obj}\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i ; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle C : C \in \text{Obj}D \rangle$.

For $X, Y \in \text{Obj}\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset \text{Obj}\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by D^\perp the class

$$\{Y \in \text{Obj}\underline{C} : X \perp Y \ \forall X \in D\}.$$

Sometimes we will denote by D^\perp the corresponding full subcategory of \underline{C} . Dually, ${}^\perp D$ is the class $\{Y \in \text{Obj}\underline{C} : Y \perp X \ \forall X \in D\}$.

We will say that C_i generate \underline{C} if \underline{C} equals $\langle C_i \rangle$. We will say that C_i *weakly generate* \underline{C} if for $X \in \text{Obj}\underline{C}$ we have $\underline{C}(C_i[j], X) = \{0\} \ \forall i \in I, j \in \mathbb{Z} \implies X = 0$ (i.e. if $\{C_i[j]\}^\perp$ contains only zero objects). Dually, C_i *weakly cogenerate* \underline{C} if ${}^\perp\{C_i[j]\} = \{0\}$.

In this paper all complexes will be cohomological i.e. the degree of all differentials is +1; respectively, we will use cohomological notation for their terms.

For an additive category A we denote by $C(A)$ the unbounded category of complexes over A ; $K(A)$ is the homotopy category of $C(A)$ i.e. morphisms of complexes are considered up to homotopy equivalence; $C^-(A)$ denotes the category of complexes over A bounded above; $C^b(A) \subset C^-(A)$ is the subcategory of bounded complexes; K^b denotes the homotopy category of bounded complexes. We will denote by $C(A)^{\leq i}$ (resp. $C(A)^{\geq i}$) the unbounded category of complexes concentrated in degrees $\leq i$ (resp. $\geq i$).

Below for a complex denoted by $\dots \rightarrow X^{-1} \rightarrow X^0 \rightarrow X^1 \rightarrow \dots$ (or

similarly) we will assume that X^i is in degree i . For other complexes we will assume (by default) that the last term specified is in degree 0.

For an abelian A we will denote by $D(A)$, $D^-(A)$, $D^b(A)$ the corresponding versions of the derived category of A .

Ab is the category of abelian groups; Ab_{fr} is the category of free abelian groups; $Ab_{fin.fr}$ is the category of finitely generated free abelian groups.

For additive C, D we denote by $\text{AddFun}(C, D)$ the category of additive functors from C to D (we will be always able to assume that C, D are small). For an additive A we will denote by A^* the category $\text{AddFun}(A, Ab)$ and by A_* the category $\text{AddFun}(A^{op}, Ab)$. Note that both of these are abelian. Moreover, Yoneda's lemma gives full embeddings of A into A_* and of A^{op} into A^* (these send $X \in \text{Obj} A$ to $X_* = A(-, X)$ and to $X^* = A(X, -)$, respectively). A'_* will denote the full abelian subcategory of A_* generated by A .

It is easily seen that any object of A is projective in A_* . Besides, any object of A_* has a resolution by (infinite) coproducts of objects of A . These facts are rather easy; the proofs can be found at the beginning of §8 of [MVW06].

The definition of a cocompact object is dual to those of a compact one: $X \in \text{Obj} \underline{C}$ is cocompact if $\underline{C}(\prod_{i \in I} Y_i, X) = \bigoplus_{i \in I} \underline{C}(Y_i, X)$ for any set I and any $Y_i \in \text{Obj} \underline{C}$ such that the product exists (we do not assume that it exists always).

We list the main definitions of this paper. Weight structures, $\underline{C}^{w \leq 0}$, $\underline{C}^{w \geq 0}$, and weight decompositions of objects are defined in 1.1.1; \underline{Hw} (the *heart* of w), $\underline{C}^{w=0}$, $\underline{C}^{w \leq l}$, $\underline{C}^{w \geq l}$, $\underline{C}^{[j, i]}$, non-degenerate, and bounded (above, below or both) weight structures are defined in Definition 1.2.1; $X^{w \leq i}$, $X^{w \geq i+1}$, $w_{\leq i} X$, and $w_{\geq i+1} X$ are defined in Remark 1.2.2; extension-stable subcategories are defined in Definition 1.3.1; \underline{C}^- , \underline{C}^+ , and \underline{C}^b are defined in Definition 1.3.5; several notation and definitions for weight decomposition of morphisms, (infinite) weight decomposition of objects, and weight Postnikov towers for objects are introduced in §1.5; the weight filtration on functors is introduced in Definition 2.1.1; weight complex of an object is defined in Definition 2.2.1; weight spectral sequences (denoted by $T(H, X)$) are introduced in §§2.3-2.4; virtual t -truncations of (co)homological functors are defined in Remark 2.5.2; $T \triangleleft \text{Mor} A$ (T is an ideal of morphisms for A) is defined in Definition 3.1.1; A/T is defined in Remark 3.1.2; the weak category of complexes $K_{\mathfrak{w}}(A)$, distinguished triangles in it, and weakly exact functors are defined in Definition 3.1.6; the weight decomposition functor WD and the weight complex functor t are introduced in Theorem 3.2.2; t -structures are recalled in §4.1; countable homotopy colimits in \underline{C} and their properties are described in §4.2; negative subcategories, Karoubi-closures, and small envelopes are introduced

in Definition 4.3.1; the categories SH and SH_{fin} of spectra are mentioned in Corollary 4.3.3; adjacent (weight and t -structures) are defined in Definition 4.4.1; weight-exact (and also t -exact) functors are defined in Definition 4.4.4; negatively well-generating sets of objects are defined in Definition 4.5.1; more categories of spectra, singular cohomology, and singular homology H^{sing} of spectra are considered in §4.6; we discuss idempotent completions in §5.1; K_0 -groups of \underline{Hw} , \underline{C} , $\text{End } \underline{Hw}$, $\text{End } \underline{C}$, $\text{End}^n \underline{Hw}$, and $\text{End}^n \underline{C}$ are defined in §§5.3–5.4; regular additive categories are defined in Definition 5.4.2; differential graded categories and twisted complexes over them are defined in §6.1; truncation functors t_N are constructed in §6.3; the spectral sequence $S(H, X)$ is considered in §6.4; we recall $SmCor$, J , \mathfrak{H} , DM_-^{eff} , DM^s , DM_{gm}^{eff} , and DM_{gm} in §6.5; recall $Corr_{rat}$, $Chow^{eff}$ and $Chow$ in §6.6; $H^i(X, \mathbb{Z}/l^n\mathbb{Z}(s))$ and $H_{et}^i(X, \mathbb{Z}/l^n\mathbb{Z}(s))$ are considered in §7.5; gluing data is defined in §8.2; weight filtration (for motives) is mentioned in §8.6.

1 Weight structures in triangulated categories: basic definitions and properties; auxiliary state- ments

In this section we give the definition of a weight structure w in a triangulated category \underline{C} (in §1.1) (this includes the notion of a *weight decomposition* of an object). We give other basic definitions and prove their certain simple properties in §1.2 and §1.3. We recall certain auxiliary statements that will help us to prove that weight decompositions are functorial (in a certain sense) in §1.4. We study weight decompositions of morphisms, *infinite weight decompositions* and weight Postnikov towers for objects in §1.5.

1.1 Weight structures: definition

Definition 1.1.1 (Definition of a weight structure). A pair of subclasses $\underline{C}^{w \leq 0}, \underline{C}^{w \geq 0} \subset \text{Obj } \underline{C}$ for a triangulated category \underline{C} will be said to define a weight structure w for \underline{C} if they satisfy the following conditions:

(i) $\underline{C}^{w \leq 0}, \underline{C}^{w \geq 0}$ are additive and Karoubi-closed (i.e. contain all retracts of their objects that belong to $\text{Obj } \underline{C}$).

(ii) **'Semi-invariance' with respect to translations.**

$$\underline{C}^{w \geq 0} \subset \underline{C}^{w \geq 0}[1], \underline{C}^{w \leq 0}[1] \subset \underline{C}^{w \leq 0}.$$

(iii) **Orthogonality.**

$\underline{C}^{w \geq 0} \perp \underline{C}^{w \leq 0}[1]$ (i.e. for any $X \in \underline{C}^{w \geq 0}$, $Y \in \underline{C}^{w \leq 0}[1]$ we have $\underline{C}(X, Y) = \{0\}$).

(iv) **Weight decompositions.**

For any $X \in \text{Obj } \underline{\mathcal{C}}$ there exists a distinguished triangle

$$B[-1] \rightarrow X \rightarrow A \xrightarrow{f} B \quad (1)$$

such that $A \in \underline{\mathcal{C}}^{w \leq 0}$, $B \in \underline{\mathcal{C}}^{w \geq 0}$.

The triangle (1) will be called a *weight decomposition* of X .

The basic example of a weight structure is given by the stupid filtration on $K(A)$ (for an arbitrary additive A ; we will call it the *stupid* weight structure). We will omit w in this case and denote by $K(A)^{\leq 0}$ (resp. $K(A)^{\geq 0}$) the class of complexes that are (homotopy equivalent to complexes) concentrated in degrees ≤ 0 (resp. ≥ 0). Its heart (see Definition 1.2.1 below) is the the Karoubization of A in $K(A)$ (so, it is equivalent to A if the latter is pseudo-abelian). Moreover, we will see below (cf. Theorems 3.2.2, 3.3.1, and Remark 3.3.4) that this example is 'almost universal' if one fixes the heart.

Note: if A is an abelian category with enough projectives and injectives, then the appropriate version (i.e. we impose some boundedness conditions) of $D(A)$ is often equivalent to $K^?(Inj A)$ and to $K^?(Proj A)$ (here $Proj A$ and $Inj A$ denote the categories of projective and injective objects of A). We obtain that some triangulated categories can support at least two distinct weight structures with non-isomorphic hearts.

Remark 1.1.2. 1. Obviously, the axioms of weight structures are self-dual (recall that the same is true for axioms of triangulated categories). This means that (C_1, C_2) define a weight structure for $\underline{\mathcal{C}}$ whenever (C_2^{op}, C_1^{op}) define a weight structure for $\underline{\mathcal{C}}^{op}$. Recall also the same is true for t -structures (see Definition 4.1.1).

2. Besides, if A is an abelian category, $F : \underline{\mathcal{C}} \rightarrow A$ is a (co)homological functor, then the same is true for the functor $F^{op} : \underline{\mathcal{C}}^{op} \rightarrow A^{op}$ obtained from F in the natural way. Hence one can interchange $\underline{\mathcal{C}}^{w \geq 0}$ with $\underline{\mathcal{C}}^{w \leq 0}$ without changing the variance of F .

We will apply these observations several times.

3. A major distinction of the axioms of weight structures from those of t -structures (see Definition 4.1.1 below) is that we have the opposite orthogonality for them; also, the arrows in t -decompositions 'go in the converse direction'. We will see below that this results in a drastic difference between the properties of these two types of structures.

1.2 Other definitions

We will also need the following definitions.

Definition 1.2.1. [Other basic definitions]

1. The category $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}^{rw=0} = \underline{C}^{rw \geq 0} \cap \underline{C}^{rw \leq 0}$, $\underline{Hw}(X, Y) = \underline{C}(X, Y)$ for $X, Y \in \underline{C}^{rw=0}$, will be called the *heart* of the weight structure w . Obviously, \underline{Hw} is additive.

2. $\underline{C}^{w \geq l}$ (resp. $\underline{C}^{w \leq l}$) will denote $\underline{C}^{rw \geq 0}[-l]$ (resp. $\underline{C}^{rw \leq 0}[-l]$).

3. For all $i, j \in \mathbb{Z}$, $i \geq j$ we define $\underline{C}^{[j, i]} = \underline{C}^{rw \geq j} \cap \underline{C}^{rw \leq i}$. By abuse of notation, we will sometimes identify $\underline{C}^{[j, i]}$ with the corresponding full additive subcategory of \underline{C} .

4. w will be called *non-degenerate* if

$$\cap_l \underline{C}^{w \geq l} = \cap_l \underline{C}^{w \leq l} = \{0\}.$$

5. w will be called *bounded above* (resp. *bounded below*) if $\cup_{l \in \mathbb{Z}} \underline{C}^{w \leq l} = \text{Obj} \underline{C}$ (resp. $\cup_{l \in \mathbb{Z}} \underline{C}^{w \geq l} = \text{Obj} \underline{C}$).

6. w will be called *bounded* if it is bounded both above and below.

Now we observe an important difference between decompositions of objects with respect to t -structures and those with respect to weight structures.

Remark 1.2.2. 1. In contrast to the t -structure situation, the presentation of X in the form (1) is (almost) never unique. The only exception is the *totally degenerate* situation when $\underline{C}^{rw=0} = \{0\}$.

Note that in this case the classes $\underline{C}^{w \leq i}$ coincide for all $i \in \mathbb{Z}$; all $\underline{C}^{w \geq i}$ coincide also. It easily follows that $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ yield (full) triangulated subcategories of \underline{C} (cf. part 3 of Proposition 1.3.3 below). In this case the weight decomposition axiom yields that the inclusion $\underline{C}^{w \leq 0} \rightarrow \underline{C}$ possesses a left adjoint. Whereas such situations are certainly important, it does not seem to make much sense to study them using the (general) formalism of weight structures. In particular, below we will mostly be interested in the non-degenerate situation (since we will 'cut' objects of \underline{C} into pieces that belong to $\underline{C}^{rw=0}$).

2. Yet we will need to choose some (A, B, f) several times. We will write that $A = X^{w \leq 0}$, $B = X^{w \geq 1}$ if there exists a distinguished triangle (1). In Theorem 3.2.2 below we will verify that $X \rightarrow (A, B, f)$ is a 'functor up to morphisms that are zero on cohomology'.

We will also often denote $(X[i])^{w \leq 0}$ by $X^{w \leq i}$ and $(X[i])^{w \geq 1}$ by $X^{w \geq i+1}$ for all $i \in \mathbb{Z}$. Note that we have $X^{w \leq i} \in \underline{C}^{rw \leq 0}$ and $X^{w \geq i+1} \in \underline{C}^{rw \geq 0}$.

Below we will introduce a similar convention for the *weight complex* of X .

Besides, we will sometimes denote $X^{w \leq i}[-i]$ by $w_{\leq i} X$ and $X^{w \geq i}[-i]$ by $w_{\geq i} X$. So, for any $i \in \mathbb{Z}$ we have a distinguished triangle

$$w_{\geq i+1} X \rightarrow X \rightarrow w_{\leq i} X.$$

Yet if X admits a weight decomposition that *avoids weight 0* (a term proposed by J. Wildeshaus, see Definition 1.6 of [Wil09c]) then the choice of such a weight decomposition for X is unique; see part 2 of Remark 1.5.2 below.

1.3 Basic properties of weight structures

We will need the following definition several times.

Definition 1.3.1. $D \subset \text{Obj}\underline{\mathcal{C}}$ will be called extension-stable if for any distinguished triangle $A \rightarrow B \rightarrow C$ in $\underline{\mathcal{C}}$ we have: $A, C \in D \implies B \in D$.

We will also say that the corresponding full subcategory is extension-stable.

Remark 1.3.2. Certainly, any extension-stable subclass of $\text{Obj}\underline{\mathcal{C}}$ is additive (i.e. closed with respect to finite direct sums) since a triangle of the form $A \rightarrow A \oplus C \rightarrow C$ is always distinguished.

For any $\underline{\mathcal{C}}, w$ the following basic properties are fulfilled. Most of these properties are parallel to those of t -structures; part 7 illustrates the distinction between these notions.

Proposition 1.3.3. 1. $\underline{\mathcal{C}}^{w \leq 0} = (\underline{\mathcal{C}}^{w \geq 1})^\perp$ (see Notation).

2. Vice versa, $\underline{\mathcal{C}}^{w \geq 0} = {}^\perp \underline{\mathcal{C}}^{w \leq -1}$.

3. $\underline{\mathcal{C}}^{w \geq 0}$, $\underline{\mathcal{C}}^{w \leq 0}$, and $\underline{\mathcal{C}}^{w=0}$ are extension-stable in the sense of Definition 1.3.1.

4. All $\underline{\mathcal{C}}^{w \leq i}$ are closed with respect to arbitrary (small) products (those, which exist in $\underline{\mathcal{C}}$).

5. All $\underline{\mathcal{C}}^{w \geq i}$ are closed with respect to arbitrary (small) coproducts (those, which exist in $\underline{\mathcal{C}}$).

6. For any weight decomposition of $X \in \underline{\mathcal{C}}^{w \geq 0}$ (see (1)) we have $A \in \underline{\mathcal{C}}^{w=0}$.

7. If $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle and $A, C \in \underline{\mathcal{C}}^{w=0}$ then $B \cong A \oplus C$.

8. If $X \in \underline{\mathcal{C}}^{w=0}$, $X[-1] \rightarrow A \rightarrow B$ is a weight decomposition (of $X[-1]$) then $B \in \underline{\mathcal{C}}^{w=0}$; $B \cong A \oplus X$.

Proof. 1. We should prove: if $\underline{\mathcal{C}}(Y, X) = \{0\}$ for some $X \in \text{Obj}\underline{\mathcal{C}}$ and all $Y \in \underline{\mathcal{C}}^{w \geq 1}$ then $X \in \underline{\mathcal{C}}^{w \leq 0}$.

Let $B[-1] \rightarrow X \rightarrow A \rightarrow B$ be a weight decomposition of X . Since $B[-1] \perp X$ we obtain that X is a retract of A ; hence $X \in \underline{\mathcal{C}}^{w \leq 0}$.

2. The proof is similar to those of part 1 and could be obtained by dualization (see Remark 1.1.2). If $B[-1] \rightarrow X[-1] \rightarrow A \rightarrow B$ is a weight decomposition of $X[-1]$ then $X[-1] \perp A$. Hence X is a retract of B .

3. Let $A, C \in \underline{C}^{w \geq 0}$. For any $Y \in \text{Obj } \underline{C}$ we have a (long) exact sequence $\cdots \rightarrow \underline{C}(C, Y) \rightarrow \underline{C}(B, Y) \rightarrow \underline{C}(A, Y) \rightarrow \cdots$; hence by part (ii) of Definition 1.1.1 we obtain that $B \perp \underline{C}^{w \leq -1}$. Now assertion 2 implies that $B \in \underline{C}^{w \geq 0}$.

The proof for the case $A, C \in \underline{C}^{w \leq 0}$ could be obtained by dualization.

The statement for the case $A, C \in \underline{C}^{w=0}$ now follows immediately from the definition of $\underline{C}^{w=0}$.

4. Obviously, assertion 1 implies that $\underline{C}^{w \leq i} = (\underline{C}^{w \geq i+1})^\perp$. This yields the result immediately.

5. Similarly, by assertion 2 we have $\underline{C}^{w \geq i} = {}^\perp \underline{C}^{w \leq i-1}$; this yields the result.

6. $A \in \underline{C}^{w \leq 0}$ by definition. Since we have a distinguished triangle $X \rightarrow A \rightarrow B \rightarrow X[1]$, assertion 3 implies that $A \in \underline{C}^{w \geq 0}$.

7. Since $C \in \underline{C}^{w \geq 0}$ and $A[1] \in \underline{C}^{w \leq -1}$, the morphism $C \rightarrow A[1]$ in the distinguished triangle is zero; so the triangle splits.

8. We have a distinguished triangle $A \rightarrow B \rightarrow X$. By assertion 3 we obtain that $B \in \underline{C}^{w=0}$. Then assertion 7 yields the result. \square

Remark 1.3.4. 1. We try to answer the questions when a morphism $b[-1] \in \underline{C}(B[-1], X)$ for $B \in \underline{C}^{w \geq 0}$ extends to a weight decomposition of X and $a \in \underline{C}(X, A)$ for $A \in \underline{C}^{w \leq 0}$ extends to a weight decomposition of X (i.e. $\text{Cone}(f) \in \underline{C}^{w \geq 0}$) using parts 1 and 2 of Proposition 1.3.3.

We apply the long exact sequence corresponding to the functor C^* for $C \in \underline{C}^{w \geq 0}$ (resp. to C_* for $C \in \underline{C}^{w \leq 0}$). In the first case we obtain that $b[-1]$ extends to a weight decomposition whenever the map $\underline{C}(C[i], B[-1]) \rightarrow \underline{C}(C[i], X)$ induced by b is bijective for $i = -2$ and is surjective for $i = -1$ for all $C \in \underline{C}^{w \geq 0}$. Dually, a extends to a weight decomposition if and only if for any $C \in \underline{C}^{w \leq 0}$ the map $\underline{C}(A, C) \rightarrow \underline{C}(X, C)$ induced by a is bijective for $i = 1$ and is injective for $i = 0$.

Moreover, in many important cases (cf. section 4 below) it suffices to check the conditions of part 1 (resp. part 2) of Proposition 1.3.3 only for $Y = C[i]$ for $C \in \underline{C}^{w=0}$, $i < 0$ (resp. for $i > 0$). Then these conditions are equivalent to the bijectivity of all maps $\underline{C}(C[i], B[-1]) \rightarrow \underline{C}(C[i], X)$ induced by b for $i < -1$ and their surjectivity for $i = -1$ for all $C \in \underline{C}^{w=0}$ (resp. to the bijectivity of all maps $\underline{C}(A, C) \rightarrow \underline{C}(X, C)$ induced by a for $i > 0$ and their injectivity for $i = 0$).

We will use this observation below.

2. Certainly, all $\underline{C}^{w \geq i}$, $\underline{C}^{w \leq i}$, and $\underline{C}^{w=i}$ are additive.

3. Since all (co)representable functors are additive, for any class of $C \subset \text{Obj } \underline{C}$ the classes C^\perp and ${}^\perp C$ are Karoubi-closed (in \underline{C}). We will use this fact below.

Definition 1.3.5. We consider $\underline{C}^- = \cup \underline{C}^{w \leq i}$ and $\underline{C}^+ = \cup \underline{C}^{w \geq i}$.

We call $\underline{C}^b = \underline{C}^+ \cap \underline{C}^-$ the class of *bounded* objects of \underline{C} .

Proposition 1.3.6. 1. \underline{C}^- , \underline{C}^+ , \underline{C}^b are Karoubi-closed triangulated subcategories of \underline{C} .

2. w induces weight structures for \underline{C}^- , \underline{C}^+ , and \underline{C}^b , whose hearts equal \underline{Hw} .

3. w is non-degenerate when restricted to \underline{C}^b .

Proof. 1. From part 3 of Proposition 1.3.3 we easily deduce that \underline{C}^- , \underline{C}^+ , \underline{C}^b are closed with respect to finite coproducts, cones of morphisms, and retracts.

2. It suffices to verify that for any object X of \underline{C}^- , \underline{C}^+ , or \underline{C}^b , the components of all of its possible weight decompositions belong to the corresponding category.

Let a distinguished triangle $B[-1] \rightarrow X \rightarrow A \rightarrow B \rightarrow X[1]$ be a weight decomposition of X , i.e $A \in \underline{C}^{w \leq 0}$, $B \in \underline{C}^{w \geq 0}$.

If $X \in \underline{C}^{w \leq i}$ for some $i > 0$ then part 3 of Proposition 1.3.3 implies $B \in \underline{C}^{w \leq i-1}$. Similarly, if $X \in \underline{C}^{w \geq i}$ for some $i \leq 0$ then $A \in \underline{C}^{w \geq i}$. We obtain the claim.

3. Let $X \in \text{Obj} \underline{C}^b \cap (\cap \underline{C}^{w \geq i})$; in particular, $X \in \underline{C}^{w \leq j}$ for some $j \in \mathbb{Z}$. Then by the orthogonality property for w we have: $X \perp X$, hence $X = 0$.

A similar argument proves that $\text{Obj} \underline{C}^b \cap (\cap \underline{C}^{w \leq i}) = \{0\}$.

□

\underline{C}^b is especially important; note that it equals \underline{C} if (\underline{C}, w) is bounded.

Now we prove a simple lemma that will help us several times below to verify that a pair of subcategories satisfy axioms of weight structures.

Lemma 1.3.7. 1. Let $C \subset \text{Obj} \underline{C}$. Then the classes $C_1 = (C^\perp)[-1]$ and $C_2 = ({}^\perp C)[1]$ are Karoubi-closed and extension-stable.

2. Let C be additive, Karoubi-closed, and satisfy $C \subset C[1]$. Suppose also that for any $X \in \text{Obj} \underline{C}$ there exist $A \in C_1$, $B \in C$, and a distinguished triangle $B[-1] \rightarrow X \rightarrow A \rightarrow B$. Then the pair (C_1, C) defines a weight structure for \underline{C} .

3. If for $D_1, D_2 \subset \text{Obj} \underline{C}$ we have $D_2 \perp D_1[1]$, then the same is true for Karoubi-closures of D_1, D_2 .

4. Let C be additive, Karoubi-closed, and satisfy $C[1] \subset C$. Suppose also that for any $X \in \text{Obj} \underline{C}$ there exist $A \in C$, $B \in C_2$ and a distinguished triangle $B[-1] \rightarrow X \rightarrow A \rightarrow B$. Then the pair (C, C_2) defines a weight structure for \underline{C} .

Proof. 1. The assertion is immediate from the fact that (co)representable functors are additive and cohomological (resp. homological); cf. the proof of part 3 of Proposition 1.3.3.

2. Applying assertion 1, we obtain: it suffices to check that $C_1[1] \subset C_1$. Now, for any $X \in C_1$, $Y \in C$ we have $\underline{C}(Y, X[2]) = \underline{C}(Y[-1], X[1]) = \{0\}$ (from the definition of C_1 and: $C[-1] \subset C \iff C \subset C[1]$).

3. Immediate from the biadditivity of $\underline{C}(-, -)$.

4. This is exactly the dual of assertion 2 (see Remark 1.1.2). \square

Lastly we prove a simple statement on comparison of weight structures.

Lemma 1.3.8. *Suppose that v, w are weight structures for \underline{C} ; let $\underline{C}^{v \leq 0} \subset \underline{C}^{w \leq 0}$ and $\underline{C}^{v \geq 0} \subset \underline{C}^{w \geq 0}$. Then $v = w$ (i.e. the inclusions are equalities).*

Proof. Let $X \in \text{Obj } \underline{C}^{w \leq 0}$; let $B[-1] \xrightarrow{h} X \rightarrow A \rightarrow B$ be a weight decomposition of X with respect to v . Since $B[-1] \in \underline{C}^{w \geq 1}$, the orthogonality property for w implies $h = 0$. Hence X is a retract of A . Since $\underline{C}^{v \leq 0}$ is Karoubi-closed, we have $X \in \underline{C}^{v \leq 0}$.

We obtain that $\underline{C}^{v \leq 0} = \underline{C}^{w \leq 0}$. The equality $\underline{C}^{v \geq 0} = \underline{C}^{w \geq 0}$ is proved similarly. \square

1.4 Some auxiliary statements: 'almost functoriality' of distinguished triangles

We will prove below that weight decompositions are functorial in a certain sense ('up to morphisms that are zero on cohomology'). We will need some (general) statements on 'almost functoriality' of distinguished triangles for this. This means that a morphism between single vertices of two distinguished triangles can often be completed to a morphism of these triangles.

Lemma 1.4.1. *Let $T : X \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} X[1]$ and $T' : X' \xrightarrow{a'} A' \xrightarrow{f'} B' \xrightarrow{b'} X'[1]$ be distinguished triangles.*

1. *Let $B \perp A'[1]$. Then for any morphism $g : X \rightarrow X'$ there exist $h : A \rightarrow A'$ and $i : B \rightarrow B'$ completing g to a morphism of triangles $T \rightarrow T'$.*

2. *Let moreover $B \perp A'$. Then g and h are unique.*

Proof. This fact can be easily deduced from Proposition 1.1.9 of [BBD82] (or Corollary IV.1.4 of [GeM03]); we use the same argument here.

1. Since the sequence $\underline{C}(B, A') \rightarrow \underline{C}(B, B') \rightarrow \underline{C}(B, X'[1]) \rightarrow \underline{C}(B, A'[1])$ is exact, there exists $i : B \rightarrow B'$ such that $b' \circ i = g[1] \circ b$. By axiom TR3 (see §IV.1 of [GeM03]) there also exist a morphism $h : A \rightarrow A'$ that completes (g, i) to a morphism of triangles.

2. Now we also have $\underline{C}(B, A') = \{0\}$. Hence the exact sequence mentioned in the proof of part I now also yields the uniqueness of i .

The condition on h is that $h \circ a = a' \circ g$. We have an exact sequence $\underline{C}(B, A') \rightarrow \underline{C}(A, A') \rightarrow \underline{C}(X, A')$. Since $B \perp A'$, we obtain that h is unique also. \square

Proposition 1.4.2. [*3 × 3-Lemma*]
Any commutative square

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ \downarrow g & & \downarrow h \\ X' & \xrightarrow{a'} & A' \end{array}$$

can be completed to a 4×4 diagram (we will mainly need its upper left 3×3 part) of the following sort:

$$\begin{array}{ccccccc} X & \xrightarrow{a} & A & \xrightarrow{f} & B & \longrightarrow & X[1] \\ \downarrow g & & \downarrow h & & \downarrow i & & \downarrow g[1] \\ X' & \xrightarrow{a'} & A' & \xrightarrow{f'} & B' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \xrightarrow{a''} & A'' & \xrightarrow{f''} & B'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \xrightarrow{a[1]} & A[1] & \xrightarrow{f[1]} & B[1] & \longrightarrow & X[2] \end{array} \quad (2)$$

such that all rows and columns are distinguished triangles and all squares are commutative, except the right lowest square which anticommutes.

Proof. The proof is mostly a repetitive use of the octahedral axiom. However it requires certain unpleasant diagrams. It is written in [BBD82], Proposition 1.1.11. \square

We will also apply the octahedral axiom (see §IV.1.1 of [GeM03]) directly. We recall that it states that any diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ can be completed to an octahedral diagram. In particular, there exists a distinguished triangle $\text{Cone}(g \circ f) \rightarrow \text{Cone}(g) \rightarrow \text{Cone}(f)[1]$, whereas the map $\text{Cone}(g) \rightarrow \text{Cone}(f)[1]$ is obtained by composing of two of the maps in the distinguished triangles that define $\text{Cone}(f)$ and $\text{Cone}(g)$ (see §IV.1.8 of [GeM03]).

In particular, we will need the following very easy application of the octahedral axiom (we will apply it for the study of Postnikov towers below).

Lemma 1.4.3. *Let $X \in \text{Obj}\underline{\mathcal{C}}$; consider a (bounded above, below or both) sequence of $\underline{\mathcal{C}}$ -morphisms $\cdots \rightarrow Y_{m-1} \rightarrow Y_m \rightarrow \dots$ equipped with morphisms $Y_i \rightarrow X$ such that all the corresponding triangles commute. Consider distinguished triangles $Y_i \rightarrow Y_{i+1} \rightarrow X_i$ for some $X_i \in \text{Obj}\underline{\mathcal{C}}$ (when the corresponding X_i are defined). Then for $Z_i = \text{Cone}(Y_i \rightarrow X)$ there also exist distinguished triangles*

$$Z_i \rightarrow Z_{i+1} \rightarrow X_i[1]. \quad (3)$$

Proof. The proof is immediate if one completes the commutative triangle $Y_i \rightarrow Y_{i+1} \rightarrow X$ to an octahedral diagram. \square

1.5 Weight decompositions of morphisms; multiple weight decompositions of objects

Starting from this moment the triangle

$$T_k[k] : X[k] \xrightarrow{a_k} X^{w \leq k} \xrightarrow{f_k} X^{w \geq k+1} \xrightarrow{b_k} X[k+1] \quad (4)$$

will be (an arbitrary choice of) a weight decomposition of $X[k]$ for some $X \in \text{Obj}\underline{\mathcal{C}}$, $k \in \mathbb{Z}$; $T'_k[k] : X'[k] \xrightarrow{a'_k} X'^{w \leq k} \xrightarrow{f'_k} X'^{w \geq k+1} \xrightarrow{b'_k} X'[k+1]$ will be a weight decomposition of $X'[k]$. Sometimes we will drop the index k in the case $k = 0$.

Lemma 1.5.1. *1. Let $l \leq m$. Then for any morphism $g : X \rightarrow X'$ there exist $h : X^{w \leq m}[-m] \rightarrow X'^{w \leq l}[-l]$ and $i : X^{w \geq m+1}[-m] \rightarrow X'^{w \geq l+1}[-l]$ completing g to a morphism of triangles $T_m \rightarrow T'_l$.*

2. Let $l < m$. Then h and i are unique.

3. For $l = m$, and any two choices of (h, i) and (h', i') as above, we have $h - h' = (s \circ f_m)[-m]$ and $i - i' = (f'_m \circ s')[-m]$ for some $s, s' \in \underline{\mathcal{C}}(X^{w \geq m+1}, X'^{w \leq m})$.

Proof. 1,2: Immediate from Proposition 1.4.1.

3. It suffices to consider the case $g, h, i = 0$. Since $a_k[-k] \circ h = 0$, T_k is a distinguished triangle, we obtain that h' can be presented as $(s \circ f_m)[-m]$. Dually, i' can be presented as $(f'_m \circ s')[-m]$ \square

Remark 1.5.2. 1. For $l < m$ we will denote i, h constructed by $g_{X^{w \leq m}, X'^{w \leq l}}$ and $g_{X^{w \geq m+1}, X'^{w \geq l+1}}$, respectively.

For $l = m = 0$ we will call any possible pair (h, i) a *weight decomposition* of g .

2. Suppose that X admits a weight decomposition that *avoids weight 0* (this notion was introduced in Definition 1.6 of [Wil09c]) i.e. a weight decomposition such that $X^{w \leq 0} \in \underline{C}^{w \leq -1}$. Then such a decomposition is unique up to a unique isomorphism. Indeed, in this case we can take $(X[-1])^{w \leq 0} = X^{w \leq 0}[-1]$. Therefore, we can apply part 2 of the previous lemma for $l = -1$, $m = 0$, $X' = X$, $g = id_X$.

Moreover, the same method yields functoriality of such weight decompositions; see Proposition 1.7. of loc.cit.

In fact, for these statements to be true it suffices to demand $\underline{Hw}(X^0, X^{-1}) = \{0\}$. In particular, this implies that weight decompositions of mixed motives are unique; cf. §8.6 below.

3. The statement of part 3 of Lemma 1.5.1 is the best possible in a certain sense. It is not possible (in general) to choose $s = s'$. In particular, one can take

$$X = X' = \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \in \text{Obj } C^{[0,1]}((\mathbb{Z}/4\mathbb{Z}) - \text{mod}) \subset C(\text{Ab}).$$

Then for $g = 0$ there exists a pair $(h, i) = (\times 2, 0)$ that is not homotopic to 0. Certainly, this example can be generalized to $X = X' = R/r^2 R$ for any commutative ring R , $r \in R$, such that $r^2 \nmid r$. In particular, this problem is not a torsion phenomenon.

Note that the example of the weight decomposition described is obviously not a 'nice' one. In particular, it cannot be extended to a 3×3 diagram. Yet adding this example to the obvious weight decomposition of id_X one obtains another weight decomposition of id_X that is not homotopy equivalent to the first one; yet it does not seem to be 'bad' in any sense.

Still one can check that extension of morphisms $(X \rightarrow X' \text{ to } (X^{w \leq i}, X^{w \geq i+1}) \rightarrow (X'^{w \leq i}, X'^{w \geq i+1}))$ via part 1 of the Lemma is sufficient to prove the functoriality of the cohomology of the weight complex of X as defined in §2.2 below (here the cohomology objects belong to \underline{Hw}'_* , see the Notation and part 2 of Remark 3.1.7 below).

We check that $g_{X^{w \leq m}, X'^{w \leq l}}$ and $g_{X^{w \geq m+1}, X'^{w \geq l+1}}$ are functorial in g and the corresponding weight decompositions.

Lemma 1.5.3. *Let $T''[j] : X''[j] \xrightarrow{a''_j} X''^{w \leq j} \xrightarrow{f''_j} X''^{w \geq j+1} \xrightarrow{b''_j} X''[j+1]$ be a weight decomposition of $X''[j]$ for $X'' \in \text{Obj } \underline{C}$ for some $j \leq i \leq 0$; let $p \in \underline{C}(X, X')$ and $q \in \underline{C}(X', X'')$.*

If $j < 0$ then for any choice of (h', h'') satisfying

$$h' \circ a_0 = a'_i[-i] \circ p \text{ and } h'' \circ a'_i[-i] = a''_j[-j] \circ q \quad (5)$$

we have $(q \circ p)_{X^{w \leq 0}, X''^{w \leq j}} = h'' \circ h'$, whereas for any choice of (i', i'') satisfying $b'_i[-i] \circ i' = p[1] \circ b_0$ and $b''_j[-j] \circ i'' = q[1] \circ b'_i[-i]$ we have $(q \circ p)_{X^{w \geq 1}, X''^{w \geq j+1}} = i'' \circ i'$.

Proof. We apply the uniqueness proved in the previous lemma.

Both sides of the first equality calculate the only map h that satisfies $h \circ a_0 = a''_j \circ (q \circ p)$, while both sides of the second equality calculate the only map i that satisfies $b''_j \circ i = (q \circ p)[1] \circ b_0$. \square

Now we prove that weight decompositions are 'exact' (in a certain sense). We relate weight decompositions of $\text{Cone}(X \rightarrow X')$ with those of X in X' ; this statement is easily seen to be related with the definition of the cone of a morphism in $C(A)$.

Lemma 1.5.4. *Let $DT : X \xrightarrow{g} X' \rightarrow C[1]$ be a distinguished triangle. Then DT can be completed to a diagram*

$$\begin{array}{ccccc} X & \xrightarrow{a_i[-i]} & w_{\leq i}X & \xrightarrow{f_i[-i]} & X^{w \geq i+1}[-i] \\ \downarrow g & & \downarrow g_{X^{w \leq i}, X'^{w \leq i-1}} & & \downarrow g_{X^{w \geq i+1}, X'^{w \geq i}} \\ X' & \xrightarrow{a'_{i-1}[1-i]} & w_{\leq i-1}X' & \xrightarrow{f'_{i-1}[1-i]} & X'^{w \geq i}[1-i] \\ \downarrow & & \downarrow & & \downarrow \\ C[1] & \longrightarrow & C_i[1-i] & \longrightarrow & C'_i[1-i] \end{array} \quad (6)$$

whose rows and columns are distinguished triangles, all squares commute, $C_i, C'_i \in \text{Obj } \underline{C}$. Moreover, the last row (shifted by $[i-1]$) gives a weight decomposition of $C[i]$.

Besides, the choice of the part of (6) consisting of six upper objects and arrows connecting them is unique (even if we don't demand that this part can be completed to the whole (6)).

Proof. By part 2 of Lemma 1.5.1, g can be uniquely completed to a morphism of triangles that are the first two rows of (6). Since the left upper square of (6) is commutative, it can be completed to a 3×3 -diagram (see Proposition 1.4.2). Hence the first two rows of this diagram will be as in (6). It remains to study the third row.

By part 3 of Proposition 1.3.3, the second column yields $C_i \in \underline{C}^{w \leq 0}$, whereas the third column yields $C'_i \in \underline{C}^{w \geq 0}$. Hence $C[i] \rightarrow C_i \rightarrow C'_i$ is a weight decomposition of $C[i]$. \square

Remark 1.5.5. 1. In fact, the lemma is valid in a more general situation. Suppose that we have a pair of full subcategories $D, E \subset \underline{C}$ that satisfy the orthogonality condition (for weight structures, i.e. $E \perp D[1]$) and are extension-stable (see Definition 1.3.1). Then any 'weight decompositions' of $X[i]$ and $X'[i-1]$ (defined similarly to the case when D, E form a weight structure) can be completed to a diagram (6) with $C_i \in \text{Obj} D$, $C'_i \in \text{Obj} E$.

Indeed, it suffices to use the orthogonality to construct the diagram required (for some $C_i, C'_i \in \text{Obj} \underline{C}$). Next, the second column yields $C_i \in \text{Obj} D$, whereas the third column yields $C'_i \in \text{Obj} E$.

We will use this statement below for constructing weight decompositions for certain 'candidate weight structures'.

2. Lemma 1.5.4 and its expansion described above show that it suffices to know weight decompositions for some basic objects of \underline{C} in order to obtain weight decompositions for all objects; see Theorems 4.3.2 and 4.5.2, and §7.1 below. The situation is quite different for t -structures; for this reason weight structures are 'more likely to exist' (than t -structures), especially in 'small' triangulated categories; cf. part 4 of Remark 4.3.4.

Now we study what happens if one combines more than one weight decomposition T_k .

Proposition 1.5.6. *[Multiple weight decompositions]*

1. *[Double weight decomposition]*

Let T_k be (arbitrary and) fixed for some $X \in \text{Obj} \underline{C}$ for k being equal to some $i, j \in \mathbb{Z}$, $i > j$.

Then there exist unique morphisms $s_{ij} : X^{w \leq i}[j-i] \rightarrow X^{w \leq j}$, $q_{ij} : X^{w \geq i+1}[j-i] \rightarrow X^{w \geq j+1}$ making the corresponding squares commutative. There also exists $X^{[i,j]} \in \underline{C}^{[0, i-j-1]}$, and distinguished triangles

$$X^{w \leq i}[j-i] \xrightarrow{s_{ij}} X^{w \leq j} \xrightarrow{c_{ij}} X^{[i,j]} \xrightarrow{d_{ij}} X^{w \leq i}[j-i+1] \quad (7)$$

and

$$X^{[i,j]}[-1] \xrightarrow{x_{ij}} X^{w \geq i+1}[j-i] \xrightarrow{q_{ij}} X^{w \geq j+1} \xrightarrow{y_{ij}} X^{[i,j]} \quad (8)$$

for some \underline{C} -morphisms $c_{ij}, d_{ij}, x_{ij}, y_{ij}$.

2. *[Infinite weight decomposition]*

Let T_k be (arbitrary and) fixed for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exist unique morphisms $s_k : X^{w \leq k}[-1] \rightarrow X^{w \leq k-1}$, $q_k : X^{w \geq k+1}[-1] \rightarrow X^{w \geq k}$

making the corresponding squares commutative. There also exist $X^k \in \underline{C}^{w=0}$, and distinguished triangles

$$X^{w \leq k}[-1] \xrightarrow{s_k} X^{w \leq k-1} \xrightarrow{c_k} X^k \xrightarrow{d_k} X^{w \leq k} \quad (9)$$

and

$$X^k[-1] \xrightarrow{x_k} X^{w \geq k+1}[-1] \xrightarrow{q_k} X^{w \geq k} \xrightarrow{y_k} X^k \quad (10)$$

for some \underline{C} -morphisms c_k, d_k, x_k, y_k .

Moreover, c_k and x_k can be chosen equal to $y_k \circ f_{k-1}$ and $(f_k \circ d_k)[-1]$, respectively.

Proof. 1. Applying Lemma 1.5.1 for $X = X'$ and $g = id_X$ we obtain the existence and uniqueness of s_{ij}, q_{ij} . It remains to study cones of these morphisms.

The 3×3 -Lemma (i.e. Proposition 1.4.2) implies that the map of distinguished triangles $T_i[i] \rightarrow T_j[j]$ can be completed to a 3×3 diagram whose rows and columns are distinguished triangles. Hence there exists a distinguished triangle $\text{Cone}(id_X) \rightarrow \text{Cone}(s_{ij}) \rightarrow \text{Cone}(q_{ij})$; therefore $\text{Cone}(s_{ij}) \cong \text{Cone}_{q_{ij}}$. Since $\underline{C}^{w \leq i-j-1}$ is extension-stable (see part 3 of Proposition 1.3.3) the distinguished triangle (7) yields $X^{[i,j]} \in \underline{C}^{w \leq i-j-1}$; the same argument applied to the distinguished triangle (8) yields $X^{[i,j]} \in \underline{C}^{w \geq 0}$.

2. The first part of the assertion is immediate from part 1 applied for $(i, j) = (k, k-1)$ for all $k \in \mathbb{Z}$.

To prove the second part it suffices to complete the commutative triangle $X[k] \xrightarrow{a_k} X^{w \leq k} \xrightarrow{s_k[1]} X^{w \leq k-1}[1]$ to an octahedral diagram.

□

Corollary 1.5.7. $\underline{C}^{w=0}$ generates \underline{C}^b (as a triangulated category).

Proof. Since \underline{C}^b is a triangulated category that contains \underline{Hw} , it suffices to prove that any object of \underline{C}^b can be obtained from objects of \underline{Hw} by a finite number of taking cones of morphisms.

Let $X \in \underline{C}^{w \geq j} \cap \underline{C}^{w \leq i}$. Then we can take $X^{w \leq k} = 0$ for $k < j$ and $X^{w \geq k} = 0$ for $k > i$ in (4). Then $X = w_{\leq i} X$ and the formula (9) gives a sequence of distinguished triangles implying that $X \in \langle \underline{C}^{w=0} \rangle$.

□

We will need the following definition several times.

Definition 1.5.8. We will denote by $Po(X)$ (a *weight Postnikov tower* for X) all the data of the (distinguished) triangles (4), (9), and (10).

Remark 1.5.9. 1. For any X, X' , and arbitrary weight Postnikov towers for them, any $g \in \underline{C}(X, X')$ can be extended to a morphism of Postnikov towers (i.e. there exist morphisms $X^{w \leq k} \rightarrow X'^{w \leq k}$, $X^{w \geq k} \rightarrow X'^{w \geq k}$, $X^k \rightarrow X'^k$ for all k , such that the corresponding squares commute). Indeed, Lemma 1.5.1 implies that we can construct morphisms $T_k \rightarrow T'_k$ desired. Next we can complete this data to morphisms of distinguished triangles of the type (9) and (10).

The main difficulty here is to prove that we can choose morphisms $X^k \rightarrow X'^k$ that will be compatible both with (9) and (10). We will not really need this fact below (since it will always be sufficient for our purposes to consider either only (9) or only (10)). So, we will only sketch the proof of this statement.

We should verify that a morphism of commutative triangles $(X[k] \rightarrow X^{w \leq k} \rightarrow X^{w \leq k-1}[1]) \rightarrow (X'[k] \rightarrow X'^{w \leq k} \rightarrow X'^{w \leq k-1}[1])$ could be completed to a morphism of the corresponding octahedral diagrams (see the proof of part 2 of Proposition 1.5.6). This is certainly true if \underline{C} has a 'reasonable model'; cf. the Corollary in §1.7 of [Mal06]. In the general case one can check: it suffices to verify that the morphisms of (9) and of (10) yield the same morphisms on the cohomology of weight complexes (of X and X' ; the target of this cohomology is $\underline{H}w'_*$; see part 2 of Remark 3.1.7 below). The latter fact could be proved using the isomorphism (16) below; see part II3 of Theorem 2.3.1 of [Bon10].

We also note here: the morphisms of weight Postnikov towers obtained this way are usually not unique; the choice of a weight Postnikov tower (for X) is not unique also. Still we will prove below that weight Postnikov towers are 'unique and functorial up to morphisms that are zero on cohomology' (in a certain sense).

The corresponding functoriality facts for cellular Postnikov towers of spectra are described in (Lemma 14 of) §6.3 of [Mar83].

2. Now, suppose that we have an unbounded system of morphisms $\dots Y_{m-1} \rightarrow Y_m \rightarrow \dots$ equipped with compatible morphisms $Y_i \rightarrow X$ (as in Lemma 1.4.3). Suppose also that $Y_l = 0$ for $l \ll 0$ and that for the corresponding X_i we have $X_i \in \underline{C}^{w=-i}$ for all $i \in \mathbb{Z}$. Then one can easily check that $Y_i \in \underline{C}^{w \geq 1-i}$. Moreover, if we also have $Y_l = X$ for $l \gg 0$, then Y_i can be chosen for $w_{\geq 1-i}X$, whereas $Z_i = \text{Cone}(Y_i \rightarrow X)$ can be taken for the corresponding $w_{\leq -i}X$.

Indeed, if $Y_l = 0$ for $l < N$ then $Y_l \in \underline{C}^{w \geq 1-l}$ for $l < N$. Now, for $l \geq N$ the distinguished triangles relating Y_i with X_j easily yield the same statement for all l (by induction on l ; here we apply part 3 of Proposition 1.3.3).

Moreover, the distinguished triangles $Y_i \rightarrow X \rightarrow Z_i$ corresponding to 'weight decompositions' exist by the definition of Z_i . Hence it suffices to

check that $Y_l = X$ for $l > N$ implies $Z_i \in \underline{\mathcal{C}}^{w \leq -i}$. We have $Z_i = 0$ for $i > N$. Hence our (last) claim can be easily deduced from the (distinguished) triangles (3).

3. More generally, the same argument as above yields that assertion 3 can be naturally extended to the case of extension-stable 'candidate weight structures' (see Remark 1.5.5).

2 Weight filtrations and spectral sequences

The goal of this section is to study the *weight spectral sequence* $T(H, X) \implies H(X)$ for $X \in \text{Obj} \underline{\mathcal{C}}$ and a (co)homological functor $H : \underline{\mathcal{C}} \rightarrow A$. T is a generalization of Deligne's weight spectral sequences (see part 2 of Remark 2.4.3), Atiyah-Hirzebruch ones (see §4.6), and (essentially) motivic descent spectral sequences of §7 of [Bon09] (cf. Remark 7.4.4 of *ibid.* and §6.4 below). So, if H is a 'classical' realization of motives then T degenerates at E_2 (rationally) and its E_2 -terms are exactly the graded pieces of the weight filtration.

In §2.1 we define the weight filtration for any functor from $\underline{\mathcal{C}}$ to an abelian category.

In §2.2 we define the *weight complex* of X (in the terms of $Po(X)$). Its 'cohomology' computes the E_2 -terms of the weight spectral sequence.

For simplicity we only consider in detail only spectral sequences for homological functors (in §2.3); dualization immediately extends the result to the cohomological functor case (see §2.4). We also note that our weight spectral sequences induce the standard weight filtration for the rational étale and Hodge realizations of varieties (and motives); we also obtain a new spectral sequence for motivic cohomology; see Remark 2.4.3.

Lastly, in §2.5 we study the D -terms of the derived exact couple i.e. D_2 for a (co)homological weight spectral sequence $T(F, X)$. There are two methods for constructing these spectral sequences (see Remark 2.3.3); the $D_2^{*,*}$ -terms (for a homological F) will be either $F'(X) = \text{Im}(F(w_{\leq k+1}(X[l])) \rightarrow F(w_{\leq k}(X[l])))$ or $F''(X) = \text{Im}(F(w_{\geq k+1}(X[l])) \rightarrow F(w_{\geq k}(X[l])))$. We prove that F', F'' are both (co)homological. For $l = 0$ the sequence $F''(X) \rightarrow F(X) \rightarrow F'(X)$ extends to a long exact sequence of functors. So, the properties of F', F'' are very similar to properties that would be satisfied by t -truncations of F if it is considered as an object of some triangulated 'category of functors' \underline{D} (so we call F', F'' *virtual t -truncations* of F). A certain explanation for this will be given in §4.4 below; see also part 2 of Remark 6.4.1 and (§2.5 of) [Bon10]. Besides, we observe that $E_2^{*,*}(T(F, X))$ can be obtained by composing two of our virtual t -truncations ('from different sides').

2.1 Weight filtration for (co)homological functors

Let A be an abelian category.

Definition 2.1.1. 1. If $H : \underline{C} \rightarrow A$ is any covariant functor then for any $i \in \mathbb{Z}$ we define $W_i(H)(X) = \text{Im}(H(w_{\geq i}(X)) \rightarrow H(X))$.

2. If $H : \underline{C} \rightarrow A$ is contravariant then we define: $W^i(H)(X) = \text{Im}(H(w_{\leq i}X) \rightarrow H(X))$.

In both cases we will call the filtration obtained the *weight filtration* of $H(X)$.

Proposition 2.1.2. 1. Let H be covariant. Then the correspondence $X \rightarrow W_i(H)(X)$ gives a canonical subfunctor of $H(X)$. This means that $W_i(H)(X)$ does not depend on the choice of a weight decomposition of $X[i]$ and for any $f : X \rightarrow Y$ we have $H(f)(W_i(H)(X)) \subset W_i(H)(Y)$ (for $X, Y \in \text{Obj}\underline{C}$).

2. The same is true for contravariant H and $W^i(H)(X)$.

Proof. 1. Part 1 of Lemma 1.5.1 implies that for any choice of weight decompositions of any $X[i], Y[i]$ we have $H(f)(W_i(H)(X)) \subset W_i(H)(Y)$. In particular, taking $Y = X$, $f = \text{id}_X$ we obtain that $W_i(H)(X)$ does not depend on the choice of the weight decomposition of $X[i]$.

2. The statement is exactly the dual of assertion 1 (see Remark 1.1.2). \square

Remark 2.1.3. 1. In the case when H is (co)homological, we can replace the images in Definition 2.1.1 by certain kernels (since they coincide).

Moreover, Proposition 2.1.2 is also true for A being **any** category with well-defined images of morphisms.

2. A partial case of this method for defining weight filtration for cohomology was (essentially) considered in Proposition 3.5 of [Han99].

3. Recall now that we have a natural embedding $i : \underline{C} \rightarrow \underline{C}_*$ that sends $X \in \text{Obj}\underline{C}$ to $X_* = \underline{C}(-, X)$. Then for any $j \in \mathbb{Z}$ and $X \in \text{Obj}\underline{C}$ we can define a functor $W_j(X)$ that sends $Y \rightarrow W_j(Y^*(X)) : \underline{C} \rightarrow \text{Ab}$. This yields an object of \underline{C}_* ; it equals $W_j(i)(X)$. We obtain a sequence of functors $W_j : \underline{C} \rightarrow \underline{C}_*$. The usual Yoneda's isomorphism $F(X) \cong \text{Mor}_{\text{AddFun}(\underline{C}, \text{Ab})}(X_*, F)$ for a contravariant functor $F : \underline{C} \rightarrow \text{Ab}$ can be easily generalized to

$$W^j(F)(X) \cong \text{Mor}_{\text{AddFun}(\underline{C}, \text{Ab})}((X_*/W_j(X)), F) \quad (11)$$

Hence the sequence of functors W_j yields a description of weight filtrations for all contravariant functors $\underline{C} \rightarrow \text{Ab}$. Moreover, for any homological $F : \underline{C} \rightarrow \text{Ab}$ one can similarly define an isomorphism

$$W_j(F)(X) \cong \text{Mor}_{\text{AddFun}(\underline{C}, \text{Ab})}(W^j(X), F) \quad (12)$$

In particular, one can apply this construction to the Chow weight filtration of Voevodsky's motives (see §6.5 below). For any motif X (and so, for any variety) one obtains a sequence of objects of $DM_{gm}^{eff,*}$ which could be called *semi-motives*. These objects contain important information on X . In particular, (11) and (12) show that they have both homological and cohomological realizations!

2.2 Weight complexes of objects (definition)

Now we describe the weight complex of $X \in \text{Obj}\underline{C}$. We will prove that it is canonic and functorial (in a certain sense) in §3.2 below.

We adopt the notation of subsection 1.5.

Definition 2.2.1. We define the morphisms $h_i : X^i \rightarrow X^{i+1}$ as $c_{i+1} \circ d_i$. We will call $t(X) = (X^i, h_i)$ the *weight complex* of X .

Note that all information on $t(X)$ is contained in $Po(X)$ (including the relation of $t(X)$ with X).

Proposition 2.2.2. 1. *Weight complex (of any $X \in \text{Obj}\underline{C}$) is a complex indeed i.e. we have $d^2 = 0$.*

2. *If for some $i \in \mathbb{Z}$ and $X \in \underline{C}^{w \leq i}$ (resp. $X \in \underline{C}^{w \geq i}$) then there exists a choice of the weight complex of X belonging to $C(\underline{Hw})^{\leq i}$ (resp. to $C(\underline{Hw})^{\geq i}$).*

Proof. 1. We have

$$h_{i+1} \circ h_i = c_{i+2} \circ (d_{i+1} \circ c_{i+1}) \circ d_i = c_{i+2} \circ 0 \circ d_i = 0$$

for all i .

2. Similarly to the proof of Corollary 1.3.6, we can take $X^{w \geq k} = 0$ for $k > i$ (resp. $X^{w \leq k} = 0$ for $k < i$). Then we would have $w_{\leq k}X = X$ for $k \geq i$ (resp. $w_{\geq k}X = X$ for $k \leq i$). Therefore the corresponding choice of the weight complex of X belongs to $C(\underline{Hw})^{\leq i}$ (resp. to $C(\underline{Hw})^{\geq i}$) by definition. \square

2.3 Weight spectral sequences for homological functors

Let A be an abelian category; let $H : \underline{C} \rightarrow A$ be a homological functor (i.e. a covariant additive functor that transfers distinguished triangles into long exact sequences). The cohomological functor case will be obtained from the homological one by dualization.

Let $X \in \text{Obj}\underline{C}$, $(X^i, h_i) = t(X)$. We construct a spectral sequence whose E_1 -terms are $H(X^i[j])$, which converges to $H(X[i+j])$ in many important cases.

Definition 2.3.1. We denote $H(Y[p])$ by $H_p(Y)$ for any $Y \in \text{Obj } \underline{C}$.

For a cohomological H we will denote by $H^p(-)$ the functor $Y \mapsto H(Y[-p])$.

First we describe the exact couple. It is obtained by applying H to the data contained in a weight Postnikov tower for X (see Definition 1.5.8). In the first three parts of Theorem 2.3.2 we will fix the choice this tower.

Our exact couple is almost the same as the couple in §IV2, Exercise 2, of [GeM03]. We take $E_1^{pq} = H_q(X^p)$, $D_1^{pq} = H_q(X^{w \geq p})$. Then the distinguished triangles (10) endow (E_1, D_1) with the structure of an exact couple.

Theorem 2.3.2. *[The homological weight spectral sequence]*

I There exists a spectral sequence $T = T(H, X)$ coming from our (E_1, D_1) with $E_1^{pq} = H_q(X^p)$ such that the map $E_1^{pq} \rightarrow E_1^{p+1q}$ equals $H_q(h_p)$.

II $T(H, X)$ converges to $H_{p+q}(X)$ in either of the following cases:

- (i) $X \in \underline{C}^b$.*
- (ii) H vanishes on $\underline{C}^{w \geq q}$ for q large enough and on $\underline{C}^{w \leq q}$ for q small enough.*
- (iii) $X \in \underline{C}^-$ (resp \underline{C}^+) and H vanishes on $\underline{C}^{w \leq q}$ for q small enough (resp. on $\underline{C}^{w \geq q}$ for q large enough).*

In all these cases the corresponding filtration on $H_(X)$ coincides with the (weight) filtration described in Definition 2.1.1.*

III T is functorial with respect to H i.e. for any transformation of functors $H \rightarrow H'$ we have a canonical morphism of spectral sequences $T(H, X) \rightarrow T(H', X)$; these morphisms respect sums and compositions of transformations.

IV T is canonical and functorial with respect to X starting from E_2 .

Proof. I These are just the standard properties of a spectral sequence coming from a Postnikov tower; see the Exercises after §IV.2 of [GeM03].

II In case (ii) $E_1(T)$ is obviously bounded.

In case (i) this will be also true if we choose the weight complex of X to be bounded (we can do this by the definition of \underline{C}^b). Now, for an arbitrary choice of the weight complex we also obtain that T will be bounded starting from E_2 by assertion IV.

The proof of boundedness in case (iii) is similar.

The connecting maps $w_{\geq p}X \rightarrow X$ also yield the connection desired of E_∞^{pq} with $H_{p+q}(X)$ (since they are compatible with (10)). Moreover, the induced filtration F_* on $H_*(X)$ is the weight filtration (of Definition 2.1.1) indeed, since

$$F_p H_{p+q}(X) = \text{Im}(D_1^{pq} \rightarrow H_{p+q}(X)) = W_p H_{p+q}(X). \quad (13)$$

III This is obvious since all the components of the exact couple are functorial with respect to H .

IV It suffices to check that the correspondence that sends X to the derived exact couple (i.e. (E_2, D_2) + the connecting morphisms) defines a functor. Now, since any $g \in \underline{C}(X, X')$ can be extended to a morphism $Po(X) \rightarrow Po(X')$ (see part 1 of Remark 1.5.9; here one can take any possible $Po(X), Po(X')$ and does not have to consider the triangles (9); the extension is not unique), we obtain that g is compatible with at least one morphism of the original couples (i.e. of $C_1 = (D_1 \rightarrow D_1 \rightarrow E_1 \rightarrow D_1)$). It remains to prove that the induced morphism of the derived couples $C_2(X) \rightarrow C_2(X')$ coming from this construction is uniquely determined by g ; hence it suffices to prove that the correspondences $X \mapsto D_2^{pq}(T)$ and $X \mapsto E_2^{pq}(T)$ define (canonical) functors (so we don't have to mind the connecting morphisms of C_2).

Now, $D_2^{pq}(T) = \text{Im}(H_{q-1}(X^{w \geq p+1}) \rightarrow H_q(X^{w \geq p}))$ is canonical and functorial by part I of Proposition 2.5.1 below. E_2 is also functorial since it can be factorized through the weight complex t (whose functoriality is checked §3.2 below), see part 3 of Remark 3.1.7; see also part 4 of Remark 2.5.2. \square

Remark 2.3.3. One can easily 'dualize' the exact couple above (see Remark 1.1.2). This means: there exists an exact couple that yields the same spectral sequence (so all E_*^{**} do not change) but with $D_1^{pq} = H_q(X^{w \leq p-1})$.

Certainly, the same observation can be applied to cohomological weight spectral sequences; see below.

2.4 Weight spectral sequences for cohomological functors; examples

Inverting the arrows in \underline{C} , we obtain the following cohomological analogue of the previous theorem.

Remark 2.4.1. If we dualize the exact couple from Theorem 2.3.2 directly, then we will obtain $D_1^{pq} = H^q(X^{w \leq -p})$. For an 'alternative' exact couple (see Remark 2.3.3) we have $D_1^{pq} = H^q(X^{w \geq 1-p})$. Yet this does not affect the spectral sequence.

Theorem 2.4.2. *[The cohomological weight spectral sequence]*

I There exists a spectral sequence $T = T(H, X)$ with $E_1^{pq} = H^q(X^{-p})$ such that the map $E_1^{pq} \rightarrow E_1^{p+1,q}$ equals $H^q(h_{-1-p})$.

II $T(H, X)$ converges to $H^{p+q}(X)$ in either of the following cases:

- (i) $X \in \underline{C}^b$.
- (ii) H vanishes on $\underline{C}^{w \geq q}$ for q large enough and on $\underline{C}^{w \leq q}$ for q small enough.

(iii) $X \in \underline{C}^-$ (resp \underline{C}^+) and H vanishes on $\underline{C}^{w \leq q}$ for q small enough (resp. on $\underline{C}^{w \geq q}$ for q large enough).

The corresponding filtration on $H^*(X)$ coincides with the weight filtration of Definition 2.1.1.

III T is functorial with respect to H i.e. for any transformation of functors $H \rightarrow H'$ we have a morphism of spectral sequences $T(H, X) \rightarrow T(H', X)$.

IV T is canonical and (contravariantly) functorial with respect to X starting from E_2 .

Proof. It suffices to apply Theorem 2.3.2 to the (homological) functor $H' : \underline{C}^{op} \rightarrow A$. \square

Remark 2.4.3. [Examples: 'classical' realizations and motivic cohomology]

1. Let w be bounded. Suppose that there are no maps between distinct weights for H i.e. there exists a family of full abelian subcategories $A_i \subset A$ such that $H^i(P) \in \text{Obj } A_i$ for all $i \in \mathbb{Z}$, $P \in \underline{C}^{w=0}$, and there are no non-zero A -morphisms between distinct A_i . Then we easily obtain that $T(H, X)$ degenerates at E_2 . Besides, for any $a \in \text{Obj } A$ there cannot exist more than one finite filtration W^j on a such that $W^j(a)/W^{j+1}(a) \in \text{Obj } A_j$; whereas our definition (2.1.1) of $W(H)(-)$ provides us with such a filtration for all $H(X)$, $X \in \text{Obj } \underline{C}$.

2. We will see in §6 below that Voevodsky's $DM_{gm}(\supset DM_{gm}^{eff})$ admits a *Chow weight structure* whose heart is $Chow(\supset Chow^{eff})$. Hence we obtain certain weight spectral sequences (that we will call *Chow-weight* ones) and weight filtrations for all realizations of motives. In particular, we have them for étale and Hodge realizations of motives, and for motivic cohomology.

Now, it is well known that for the rational étale and Hodge realizations there are no non-zero morphisms between distinct weights (in the corresponding categories of mixed structures) in the sense described above. Therefore for the rational étale and Hodge realizations of motives our weight filtration coincides with the usual one (up to a shift of indices; see also §7.4 of [Bon09]).

Recall also that 'classically' the weight filtration is well-defined only for cohomology with rational coefficients. Yet our method allows to define canonical weight filtrations integrally; this generalizes the construction of Theorem 3 of [GiS96].

Now we consider the case of motivic cohomology. A simple example of the spectral sequence obtained comes from the Bloch's long exact localization sequence for higher Chow groups of varieties; cf. part 1 of Remark 7.3.1 in [Bon09]. Recall that it relates motivic cohomology of $X \setminus Z$ with those of X and Z , where Z, X are smooth, Z is closed in X . In the motivic setting it comes from the Gysin distinguished triangle: $M_{gm}(X \setminus Z) \cong$

$\text{Cone}(M_{gm}(X) \rightarrow M_{gm}(Z)(c)[2c])[-1]$ (c is the codimension of Z ; see §6.5 below and Proposition 5.21 of [Deg08]). Now, suppose additionally that X is projective (so Z also is). Then $M_{gm}(X)$ and $M_{gm}(Z)(c)[2c]$ are Chow motives; so they belong to $DM_{gm}^{eff, w_{Chow}=0}$ (see §6.6). Since motivic cohomology is a (representable) cohomological functor on DM_{gm}^{eff} , we obtain: the Chow-weight spectral sequence converging to motivic cohomology of $X \setminus Z$ (corresponding to w_{Chow}) reduces to the Bloch's long exact sequence. Since the latter is non-trivial in general, the Chow-weight spectral sequences obtained is non-trivial either; it appears not to be mentioned in the literature. This filtration is compatible with regulator maps (whose targets are classical cohomology theories). Unfortunately, morphisms of motivic cohomology of motives (induced by DM_{gm}^{eff} -morphisms) are not necessarily strictly compatible with the weight filtration for this theory (in contrast with the properties of the weight filtration for rational singular and étale cohomology).

2.5 Higher D -terms of exact couples; virtual t -truncations for (co)homological functors

Now we study the higher D -terms of exact couples (i.e. the D -terms for derived exact couples) and especially D_2 for (co)homological weight spectral sequences, in more detail.

Let A be an abelian category; let $j > 0$, $k \in \mathbb{Z}$, be fixed.

Proposition 2.5.1. *I Let $F : \underline{C} \rightarrow A$ be a covariant functor. Then the assignments $F_1 = F_1^{kj} : X \mapsto \text{Im}(F(w_{\leq k+j}X) \rightarrow F(w_{\leq k}X))$ and $F_2 = F_2^{kj} : X \mapsto \text{Im}(F(w_{\geq k+j}X) \rightarrow F(w_{\geq k}X))$ define functors $\underline{C} \rightarrow A$ that do not depend (up to a canonical isomorphism) from the choice of weight decompositions. Besides, there exist natural transformations $F_2 \rightarrow F \rightarrow F_1$.*

II Let $F : \underline{C} \rightarrow A$ be homological (covariant); let $j = 1$. Then the following statements are valid.

1. F_l ($l = 1, 2$) are also homological.
2. The natural transformations $F_2 \rightarrow F \rightarrow F_1$ extend canonically to a complex of functors $\cdots \rightarrow F_1 \circ [-1] \rightarrow F_2 \rightarrow F \rightarrow F_1 \rightarrow F_2 \circ [1] \rightarrow \cdots$ that is long exact when applied to any $X \in \text{Obj} \underline{C}$.

III. Let $F : \underline{C} \rightarrow A$ be a contravariant functor.

1. The assignments $F_1 = F_1^{kj} : X \mapsto \text{Im}(F(w_{\leq k}X) \rightarrow F(w_{\leq k+j}X))$ and $F_2 = F_2^{kj} : X \mapsto \text{Im}(F(w_{\geq k}X) \rightarrow F(w_{\geq k+j}X))$ define contravariant functors $\underline{C} \rightarrow \text{Ab}$ that do not depend (up to a canonical isomorphism) from the choice of weight decompositions. There exist natural transformations $F_1 \rightarrow F \rightarrow F_2$.

2. If F is cohomological, $j = 1$, then F_l ($l = 1, 2$) also are; the transformations $F_1 \rightarrow F \rightarrow F_2$ extend canonically to a long exact sequence of

functors $\cdots \rightarrow F_2 \circ [1] \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow F_1 \circ [-1] \rightarrow \cdots$ (i.e. the sequence is exact when applied to any $X \in \text{Obj} \underline{\mathcal{C}}$).

Proof. I We use a very simple observation: for any commutative square in A

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow \\ Z & \xrightarrow{g} & T \end{array}$$

if we fix the rows then the morphism $g \circ h : X \rightarrow T$ completely determines the morphism $\text{Im } f \rightarrow \text{Im } g$ induced by h .

It follows: Lemma 1.5.3 easily implies that both F_l ($l = 1, 2$) are well-defined and functorial. Indeed, by this lemma the morphisms $w_{\leq k+j} X \rightarrow w_{\leq k} X'$ and $w_{\geq k+j} X \rightarrow w_{\geq k} X'$ 'induced' by $g \in \underline{\mathcal{C}}(X, X')$ are compatible with arbitrary morphisms of weight decompositions of X and X' that come from morphisms of objects.

It follows that fixing arbitrary weight decompositions for all $X \in \text{Obj} \underline{\mathcal{C}}$ one obtains $F_1(X)$, $F_2(X)$, and functorial connecting morphisms $F_1(X) \rightarrow F_1(X')$, $F_2(X) \rightarrow F_2(X')$ for $g \in \underline{\mathcal{C}}(X, X')$, that depend on the choices made only up to a canonical isomorphism.

It remains to note that the connecting maps $w_{\leq k+j} X \rightarrow w_{\leq k} X'$ and $w_{\geq k+j} X \rightarrow w_{\geq k} X'$ were chosen to be compatible with $w_{\geq k} X \rightarrow X \rightarrow w_{\leq k+j} X$; this yields the existence of the transformations in question.

II 1. Since F is additive, both F_l also are.

Now we check that F_1 is homological. It suffices to check that for any distinguished triangle $U = (C \rightarrow X \rightarrow X')$ the sequence $F_1(X'[-1]) \rightarrow F_1(C) \rightarrow F_1(X)$ is half-exact (i.e. exact in the middle term). Note that this sequence is obviously a complex (since the composition of morphisms is zero, F_1 of them also is).

Note also that we can use arbitrary choices of weight decompositions (by assertion I). We fix weight decompositions of X, X' (shifted) arbitrarily and look for a 'nice' weight decomposition of C .

We use the notation of §1.5. In particular, we easily obtain that $F_1(X) = \text{Ker}(F(w_{\leq k} X) \rightarrow F(X^{k+1}[-k]))$ and $F_1(X'[-1]) = \text{Ker}(F(w_{\leq k-1}(X')[-1]) \rightarrow F(X'^k[-k]))$. Our goal is to find a weight decomposition of C such that there will be a distinguished triangle $w_{\leq k-1}(X')[-1] \rightarrow w_{\leq k} C \rightarrow w_{\leq k} X$ compatible with U and $F_1(C) = \text{Ker}(F(w_{\leq k} C) \rightarrow F((X^{k+1} \oplus X'^k)[-k]))$.

We apply Lemma 1.5.4 for $i = k, k+1$. By the lemma, the triangles $C[i] \rightarrow C_i \rightarrow C'_i$ obtained from (6) by shifting the last row are weight decompositions of $C[i]$ (for all $i \in \mathbb{Z}$).

Next, we apply Lemma 1.5.4 to the morphism $g_{X^{w \leq k+1}, X'^{w \leq k}}[k+1]$ and the weight decompositions $X^{k+1} \rightarrow X^{w \leq k+1} \rightarrow X^{w \leq k}[1]$ and $X'^k[1] \rightarrow X'^{w \leq k}[1] \rightarrow X'^{w \leq k-1}[2]$ of the corresponding objects. We obtain a diagram

$$\begin{array}{ccccc}
X^{k+1} & \longrightarrow & X^{w \leq k+1} & \longrightarrow & X^{w \leq k}[1] \\
\downarrow & & \downarrow & & \downarrow \\
X'^k[1] & \longrightarrow & X'^{w \leq k}[1] & \longrightarrow & X'^{w \leq k-1}[2] \\
\downarrow & & \downarrow & & \downarrow \\
D_{k+1}[1] & \longrightarrow & C_{k+1}[1] & \xrightarrow{t[2]} & C_k[2]
\end{array} \tag{14}$$

for some $D_{k+1} \in \text{Obj} \underline{C}$ and some t . The first column gives $D_{k+1} \cong X^{k+1} \oplus X'^k$. Indeed, $X^{k+1}, X'^k \in \underline{C}^{w=0}$, whereas $\underline{C}^{w=0}$ is extension-stable by part 3 of Proposition 1.3.3, and any extension in \underline{Hw} splits by part 7 of loc.cit. Besides, t equals the morphism connecting the weight decompositions of $C[i]$ for $i = k, k+1$, since this morphism is unique by part 2 of Lemma 1.5.1. So, we have achieved our goal.

Now, we have a short exact sequence $F(w_{\leq k-1}(X')[-1]) \rightarrow F(w_{\leq k}C) \rightarrow F(w_{\leq k}X)$. Therefore, if $x \in F_1(C)$ vanishes in $F_1(X)$, we obtain that it comes from some $y \in F(w_{\leq k-1}(X')[-1])$. Hence it suffices to check that the image of y vanishes in $F((X'^k[-k]))$. Since the image of x in $F((X'^k \oplus X^{k+1})[-k])$ vanishes, it suffices to note that $F(X'^k[-k]) \hookrightarrow F((X'^k \oplus X^{k+1})[-k])$. Certainly, this reasoning could have been written down without the use of 'elements' (of objects of A).

F_2 is homological for similar reasons; this fact also immediately follows from the previous one by part 2 of Remark 1.1.2.

2. Let weight decompositions of all $X[i]$ be fixed.

The exactness of

$$\begin{aligned}
F_2(X) &= \text{Im}(F(w_{\geq k+1}(X)) \rightarrow F(w_{\geq k}X)) \rightarrow F(X) \\
&\rightarrow F_1(X) = \text{Im}(F(w_{\leq k+1}X) \rightarrow F(w_{\leq k}X))
\end{aligned}$$

in $F(X)$ is immediate from the exactness of $F(w_{\geq k+1}X) \rightarrow F(X) \rightarrow F(w_{\leq k}X)$ (in the middle).

Next, by part 2 of Proposition 1.5.1 we obtain that id_X yields canonically

a diagram

$$\begin{array}{ccccccc}
F(w_{\geq k+2}X) & \longrightarrow & F(X) & \longrightarrow & F(w_{\leq k+1}X) & \longrightarrow & F((w_{\geq k+2}X)[1]) \\
\downarrow & & \downarrow F(id_X) & & \downarrow & & \downarrow \\
F(w_{\geq k+1}X) & \longrightarrow & F(X) & \longrightarrow & F(w_{\leq k}X) & \longrightarrow & F((w_{\geq k+1}X)[1]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(X^{k+1}[-1-k]) & \longrightarrow & 0 & \longrightarrow & F(X^{k+1}[-k]) & \longrightarrow & F(X^{k+1}[-k]) \\
& & & & & & (15)
\end{array}$$

the rows and columns are exact in non-edge terms.

Hence we obtain a well-defined functorial morphism

$$F_1(X) \rightarrow \text{Im}(F((w_{\geq k+2}X)[1]) \rightarrow F((w_{\geq k+1}X)[1])) = F_2(X[1]).$$

This map will be our boundary morphism (of functors).

Now, to check the exactness of the complex $F(X) \rightarrow F_1(X) \rightarrow F_2(X[1])$ in the middle it suffices to check the inclusion:

$$\begin{aligned}
& \text{Coker}(F(w_{\leq k+1}X) \rightarrow F(w_{\leq k}X)) = \text{Ker}(F(w_{\leq k}X) \rightarrow F(X^{k+1}[-k])) \\
& \hookrightarrow \text{Coker}(F(w_{\geq k+2}X[1]) \rightarrow F(w_{\geq k+1}X[1])) = \text{Ker}(F(w_{\geq k+1}X[1]) \rightarrow F(X^{k+1}[-k]));
\end{aligned}$$

cf. the proof of assertion III1 (above). Hence the last row of (15) yields the result.

Lastly, the exactness of $F_1(X[-1]) \rightarrow F_2(X) \rightarrow F(X)$ in the middle follows easily from the previous exactness by part 2 of Remark 1.1.2.

III These statements are exactly the duals of those of parts I,II; see (part 1 of) Remark 1.1.2. \square

Remark 2.5.2. [virtual t -truncations of F]

1. For a (co)homological F we will call F_l^{k1} , $l = 1, 2$, $k \in \mathbb{Z}$, *virtual t -truncations* of F . Note that F often lies in a certain triangulated 'category of (co)homological functors' \underline{D} (of functors $\underline{C} \rightarrow A$). The simplest case is when we consider $\underline{D} \supset \underline{C}$ (or $= \underline{C}$) and F is the restriction to \underline{C} of the functor (co)represented by some $Y \in \text{Obj} \underline{D}$; see also part 2 of Remark 6.4.1. If such a \underline{D} exists, then (see parts 7,8 of Theorem 4.4.2, and part 5 of Remark 4.4.3 for the general case) the virtual t -truncations defined are often actual t -truncations of F (corresponding to a certain t -structure t on \underline{D}). In the case when $\underline{C} = \underline{D}$ this t -structure is called *adjacent* to w ; see Definition 4.4.1.

Still, it is very amusing that these t -truncated functors as well as their transformations corresponding to t -decompositions (see Definition 4.1.1) can be defined without specifying any particular \underline{D} !

2. In particular, for any $Y \in \text{Obj}SH$ the functors (co)represented by t -truncations of Y with respect to the Postnikov t -structure are exactly our F_l^{k1} (for $l = 1, 2$, $k \in \mathbb{Z}$; here F is either $SH(Y, -)$ or $SH(-, Y)$; see §4.6 for the definition of t_{Post}). Hence one can express the restrictions of F_l^{k1} to SH_{fin} in terms of F restricted to SH_{fin} (i.e. without considering infinite spectra). Note that one can obtain an Eilenberg-MacLane spectrum for \mathbb{Z} by considering the zeroth Postnikov 't-cohomology' of S^0 .

A similar observation can be applied to the pair $DM_{gm}^{eff} \subset DM_-^{eff}$ (see §7.1 below). So, though the *Chow t-truncations* of geometric motives are (usually) non-geometric, we can express functors represented by them in terms of morphisms of DM_{gm}^{eff} .

3. $F_2^{k1}(X[p])$ defined in part I of the proposition, yield the D -terms of the derived couple for the exact couple used for the proof of Theorem 2.3.2 (for a homological F). By Remark 2.4.1, for a cohomological F the corresponding D -terms are given by $F_1^{k1}(X[p])$ (defined in part III of the proposition).

Recall also, that making the alternative choice of exact couples interchanges the roles of F_l^{k1} ($l = 1, 2$) here, without changing the spectral sequences (see Remark 2.3.3). Moreover, for any $j \geq 1$ the functors $F_l^{jk}(X[p])$ ($l = 1, 2$) yield $D_{j+1}(T)$ (i.e. they calculate the D -terms of two possible choices of higher derived exact couples).

4. The definition implies that any two of our virtual t -truncation operations (for different k, l) commute. Besides for a homological F one can easily check:

$$(F_1^{1,0})_2^{1,-1} = \text{Im}(F(X^0 \rightarrow X^1) \rightarrow F(X^{-1} \rightarrow X^0)) \cong E_2^{0,0}(T(F, X)); \quad (16)$$

a similar fact is also valid in the case when F is cohomological. Thus the terms $E_2^{*,*}(T(F, X))$ of the weight spectral sequence can be expressed in terms of our virtual t -truncations; in particular, they are given by well-defined (co)homological functors. We prove that this approach yields a full description of the derived exact couple for T in (part II of) Theorem 2.4.2 of [Bon10].

5. The remarks above can be vastly extended (see §§2.3–2.6 of [Bon10]). Let there be a (*nice*) *duality* $\Phi : \underline{C}^{op} \times \underline{D} \rightarrow A$ (see part 5 of Remark 4.4.3 below and §2.5 of [Bon10]); let a t -structure t for \underline{D} be *orthogonal* to w (for \underline{C}) with respect to Φ . In this case the virtual t -truncations of functors of the type $\Phi(-, Y)$, $Y \in \text{Obj}\underline{D}$ are exactly the functors 'represented via Φ ' by the actual t -truncations of Y (corresponding to t ; see Proposition 2.5.4 of *ibid.*). This allows to establish a natural isomorphism for the weight spectral sequence for $\Phi(-, Y)$ to the one coming from t -truncations of Y (in Theorem 2.6.1 of *ibid.*).

Moreover, one can give a certain 'axiomatic' description of virtual t -truncations; see Theorems 2.3.1 and 2.3.5 of *ibid.*

3 The weight complex functor

In §5 of [Bon09] for a triangulated category $\underline{\mathcal{C}}$ with a *negative differential graded enhancement* (this includes Voevodsky's motives) an exact conservative *weight complex* functor $t_0 : \underline{\mathcal{C}} \rightarrow K(\underline{Hw})$ (in our notation) was constructed. The goal of this section is to extend this result to the case of arbitrary $(\underline{\mathcal{C}}, w)$. A reader only interested in motives could skip this section (since a 'stronger' version of the weight complex functor for all 'enhanceable' categories will be constructed in §6.3 below).

As shown in Remark 1.5.9, any $g : X \rightarrow X'$ where $X, X' \in \text{Obj} \underline{\mathcal{C}}$, can be extended to a morphism of (any possible) weight Postnikov towers for X, X' . Moreover, for compositions of g 's the corresponding morphisms of weight Postnikov towers can be composed. Yet, as the example of part 3 of Remark 1.5.2 shows, this construction cannot give a canonical morphism of weight complexes in $K(\underline{Hw})$. We have to consider a certain factor $K_{\mathfrak{w}}(\underline{Hw})$ of this category. This factor is no longer triangulated (in the general case; yet cf. Remark 3.3.4). Still the kernel of the projection $K(\underline{Hw}) \rightarrow K_{\mathfrak{w}}(\underline{Hw})$ is an ideal (of morphisms) whose square is zero; so our (weak) weight complex functor is not much worse than the 'strong' one of [Bon09].

We define and study $K_{\mathfrak{w}}(\underline{Hw})$ in §3.1. We construct the weight complex functor t in §3.2 and prove its main properties in §3.3. One of our main tools is the *weight decomposition functor* $WD : \underline{\mathcal{C}} \rightarrow K_{\mathfrak{w}}^{[0,1]}(\underline{\mathcal{C}})$; see Theorem 3.2.2.

One of the main properties of the functor t is that it calculates the E_2 -terms of the weight spectral sequence T , see part 3 of Remark 3.1.7. In fact, this is why t is called the weight complex; this term was used for the first time in [GiS96] (see §2 and §3.1 of *ibid.*).

3.1 The weak category of complexes

Let A be an additive category. We will need the following, very natural definition. The author would like to note that this definition, as well as several related (and interesting) definitions and results were also independently introduced in [Pir07].

Definition 3.1.1. A class T of morphisms in A will be called a (two-sided) ideal if it is closed with respect to sums and differences (of two morphisms of T lying in the same morphism group), finite direct sums, and compositions with any morphisms of A .

We will abbreviate these properties as $T \triangleleft Mor A$.

Remark 3.1.2. For any $T \triangleleft Mor A$ we can consider an additive category A/T whose object are the same as for A , and $A/T(X, Y) = A(X, Y)/T(X, Y)$ for all $X, Y \in Obj A$.

Besides, it is easily seen that one can naturally 'multiply' ideals of $Mor A$ via the composition operation.

Now, we will denote by $Z(X, Y)$ for $X, Y \in Obj K(A)$ the subgroup of $K(A)(X, Y)$ consisting of morphisms that can be presented as $(s_{i+1} \circ d_X^i + d_Y^{i-1} \circ t_i)$ for some set of $s_i, t_i \in A(X^i, Y^{i-1})$ (here $X = (X^i)$, $Y = (Y^i)$).

Remark 3.1.3. We will often use the fact that $sd + dt = (s - t)d + (dt + td)$ is homotopy equivalent to $(s - t)d$; hence we may assume that $t = 0$ in the definition of Z .

Now we check that for $Z = \cup_{X, Y \in Obj K(A)} Z(X, Y)$ we have $Z \triangleleft Mor K(A)$ and $Z^2 = 0$. A easy standard argument also shows that for any \underline{C} all ideals $Z \triangleleft Mor \underline{C}$ that satisfy $Z^2 = 0$ also possess a collection of nice properties.

Lemma 3.1.4. *I1. $Z \triangleleft Mor K(A)$.*

2. Let $L, M, N \in Obj K(A)$, let $g \in Z(L, M) \subset K(A)(L, M)$, $h \in Z(M, N) \subset K(A)(M, N)$. Then $h \circ g = 0$ (in $K(A)$).

II Let $T \triangleleft Mor \underline{C}$ for some additive category \underline{C} , suppose also that $T^2 = 0$; let D be an additive category. Let $p : \underline{C} \rightarrow D$ be an additive functor such that for any $X, Y \in Obj \underline{C}$ we have $\text{Ker}(\underline{C}(X, Y) \rightarrow D(p(X), p(Y))) = T(X, Y)$.

Then the following statements are valid.

1. Let p be a full functor. Then it is conservative i.e. $p(g)$ is an isomorphism whenever g is (for any morphism g in \underline{C}).

2. For any $X \in Obj \underline{C}$ and $r \in \underline{C}(X, X)$, if $p(r)$ is an idempotent then it can be lifted to an idempotent $r' \in \underline{C}(X, X)$ (i.e. $p(r') = p(r)$).

3. If \underline{C} is idempotent complete then its categorical image in D also is. Here we consider a not necessarily full subcategory of D such that all of its objects and morphisms are exactly those that come from \underline{C} .

Proof. I1. Obviously, Z is closed with respect to sums and (finite) coproducts.

Lastly, let d denote the differential, let f, g , and h be composable morphisms; let $g = s \circ d$, for s being a collection of arrows shifting the degree by -1 . Then we have $f \circ g = (f \circ s) \circ d$ and $g \circ h = -(s \circ h) \circ d$; note that h 'anticommutes with the differential'.

2. Let $L = (L_i)$, $M = (M_i)$, $N = (N_i)$. Suppose that for all $i \in \mathbb{Z}$ we have $g_i = s_{i+1} \circ d_L^i$ for some set of $s_i \in A(L_i, M_{i-1})$, whereas $h_i = u_{i+1} \circ d_M^i$ for some set of $u_i \in A(M_i, N_{i-1})$

Then $h_i \circ g_i = u_{i+1} \circ d_M^i \circ s_{i+1} \circ d_L^i$. Recall now that g is a morphism of complexes; hence for all $i \in \mathbb{Z}$ we have $d_M^i \circ s_{i+1} \circ d_L^i = d_M^i \circ d_M^{i-1} \circ s_i = 0$. We obtain that $h \circ g$ is homotopic to 0.

II1. Since p is a functor, it sends isomorphisms to isomorphisms.

Now we prove the converse statement. Let $g \in \underline{C}(X, X')$ for $X, X' \in \text{Obj} \underline{C}$, let $p(h)$ for some $h \in \underline{C}(X', X)$ be the inverse to $p(g)$. We have $h \circ g - id_X \in T(X, X)$ and $g \circ h - id_{X'} \in T(X', X')$. It suffices to check that $h \circ g$ and $g \circ h$ are invertible in \underline{C} . The last assertion follows from equalities $(h \circ g - id_X)^2 = 0$ and $(g \circ h - id_{X'})^2 = 0$ in \underline{C} , that yield $(h \circ g)(2id_X - h \circ g) = id_X$ and $(g \circ h)(2id_{X'} - g \circ h) = id_{X'}$.

2. This is just the standard statement that idempotents can be lifted (in rings).

We consider $r' = -2r^3 + 3r^2$. Since $p(r)^2 = p(r)$ in D and $r' = r + (r^2 - r) \circ (id_X - 2r)$, we have $p(r') = p(r)$. Since $r_i'^2 - r' = (r^2 - r)^2 \circ (4r^2 - 4r - 3id_X)$, we obtain that r' is an idempotent.

3. The assertion follows immediately from II2. Indeed, any idempotent d in the image can be lifted to an idempotent c in \underline{C} . Since c splits in \underline{C} , $p(c) = d$ splits in the image.

□

Remark 3.1.5. The assertions of part II remain valid for any nilpotent T . For l that satisfies $l^n = 0$, $n > 0$, the inverse to $id_X - l$ is given by $id_X + l + l^2 + \dots + l^{n-1}$. If $l^n = 0$, $r^2 - r = l$, then the equality $(x - (x - 1))^{2n-1} = 0$ allows to construct explicitly a polynomial $P(x)$ such that $P \equiv 0 \pmod{x^n \mathbb{Z}[x]}$ and $P \equiv 1 \pmod{(x - 1)^n \mathbb{Z}[x]}$. Then $P(r)^2 = P(r)$; $P(r) - r$ can be factorized through l .

Definition 3.1.6. [The definition of $K_{\mathfrak{w}}(A)$]

We define $K_{\mathfrak{w}}(A)$ as $K(A)/Z$ (in the sense of Remark 3.1.2) with isomorphic objects (i.e. homotopy equivalent complexes) identified.

We have the obvious shift functor $[1] : K_{\mathfrak{w}}(A) \rightarrow K_{\mathfrak{w}}(A)$.

A triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $K_{\mathfrak{w}}(A)$ will be called distinguished if any of its two sides can be lifted to two sides of some distinguished triangle in $K(A)$.

An additive functor $F : \underline{C} \rightarrow K_{\mathfrak{w}}(A)$ for a triangulated \underline{C} will be called *weakly exact* if it commutes with shifts and sends distinguished triangles to distinguished triangles.

The bounded subcategories of $K_{\mathfrak{w}}(A)$ are defined in the obvious way.

Remark 3.1.7. [Why $K_{\mathfrak{w}}(A)$ is a category; cohomology]

1. $K_{\mathfrak{w}}(A)$ is a category since we just factorize the class of objects of $K(A)/Z$ with respect to a class of invertible morphisms; see Remark 3.1.2.

2. Let B be an abelian category; let $F : A \rightarrow B$ be an additive functor. Then any $g \in Z(X, Y)$ gives a zero morphism on cohomology of $F_*(X)$. It follows that the cohomology of $F_*(X)$ gives well-defined functors $K_{\mathfrak{w}}(A) \rightarrow B$. Besides, these functors are easily seen to be cohomological i.e. they translate distinguished triangles in $K_{\mathfrak{w}}(A)$ into long exact sequences.

In particular, this is true for the 'universal' functor $A \rightarrow A'_*$ (recall that A'_* is the full abelian subcategory of A_* generated by A). Hence there are well-defined cohomology functors $H_i : K_{\mathfrak{w}}(A) \rightarrow A'_*$.

3. Now suppose that for a triangulated \underline{C} we have a weakly exact functor $u : \underline{C} \rightarrow K_{\mathfrak{w}}(A)$. Then the cohomology of $F_*(u(X))$ gives well-defined functors $\underline{C} \rightarrow B$. Again, distinguished triangles in \underline{C} become long exact sequences.

In particular, this statement can be applied to the weight complex $t : \underline{C} \rightarrow K_{\mathfrak{w}}(\underline{Hw})$ (whose functoriality is proved in part II of Theorem 3.2.2 below). This concludes the proof of (part IV of) Theorem 2.3.2.

Lemma 3.1.4 immediately yields the following statement.

Proposition 3.1.8. *1. The projection $p : K(A) \rightarrow K_{\mathfrak{w}}(A)$ is conservative.*

2. Let A be idempotent complete. Then $K_{\mathfrak{w}}^b(A)$ is idempotent complete also.

Proof. 1. Immediate from part II1 of Lemma 3.1.4.

2. It is well known that $K^b(A)$ is idempotent complete; see, for example, [BaS01]. Hence part II3 of Lemma 3.1.4 yields the result. \square

3.2 The functoriality of the weight complex

We will use the following simple fact.

Lemma 3.2.1. *If $X \in \underline{C}^{w \geq 0}$, $Y \in \underline{C}^{w \leq 0}$, then any $f \in \underline{C}(X, Y)$ can be factorized through some morphism $X^0 \rightarrow Y^0$ (of the zeroth terms of weight complexes).*

Proof. Easy from the equality $\underline{C}(X^{w \geq 1}[-1], Y) = \underline{C}(X^0, Y^{w \leq -1}[1]) = \{0\}$. \square

Now we prove that the usual and 'infinite' weight decompositions define certain functors. Let X, X' denote arbitrary objects of \underline{C} .

Theorem 3.2.2. *I1. The (single) weight decomposition of objects and morphisms gives a functor $WD : \underline{C} \rightarrow K_{\mathfrak{w}}^{[0,1]}(\underline{C})$ (certainly, here we only obtain two-term complexes, and we put them in degrees $[0, 1]$).*

2. Morphisms $g \in \underline{C}(X, X')$, $h \in \underline{C}(X^{w \leq 0}, X'^{w \leq 0})$ and $i : \underline{C}(X^{w \geq 1}, X'^{w \geq 1})$ give a morphism of weight decompositions (of X and X') if and only if $(h, i) = WD(g)$ in $K_{\mathfrak{w}}(\underline{C})$.

3. The homomorphism $\underline{C}(X, X') \rightarrow K_{\mathfrak{w}}^{[0,1]}(\underline{C})(WD(X), WD(X'))$ is surjective.

4. For all $X, X' \in \text{Obj} \underline{C}$ we make the notation

$$T(X, X') = \text{Ker}(\underline{C}(X, X') \rightarrow K_{\mathfrak{w}}(\underline{Hw})(WD(X), WD(X'))).$$

Then $T \triangleleft \text{Mor} \underline{C}$; $T^2 = 0$.

5. If $WD(X) \cong WD(X')$ in $K_{\mathfrak{w}}(\underline{C})$ then $X \cong X'$ in \underline{C} .

6. For any $X \in \text{Obj} \underline{C}$, $p \in \underline{C}(X, X)$, if $WD(p)$ is idempotent then $WD(p)$ can be lifted to an idempotent $p' \in \underline{C}(X, X)$.

II The correspondence $X \rightarrow t(X)$ gives a functor $\underline{C} \rightarrow K_{\mathfrak{w}}(\underline{Hw})$.

Proof. 1. By part 1 of Lemma 1.5.1, any morphism $X \rightarrow X'$ can be extended to a morphism of their (fixed) weight decompositions. This extension is uniquely defined in $K_{\mathfrak{w}}^{[0,1]}(\underline{C})$ by part 3 of loc. cit. One can compose such homomorphisms in $K_{\mathfrak{w}}(\underline{C})$ since one of the possible extensions of the composition of morphisms $X \rightarrow X' \rightarrow X''$ (in \underline{C}) is the composition of (arbitrary) extensions for the morphisms $X \rightarrow X'$ and $X' \rightarrow X''$.

It remains to check that the image of X in $\text{Obj} K_{\mathfrak{w}}^{[0,1]}(\underline{C})$ does not depend on the choice of the weight decomposition. Let $K, K' \in \text{Obj} K(\underline{C})$ be given by two weight decompositions of X ; id_X induces $g \in K(\underline{C})(K, K')$ and $h \in K(\underline{C})(K', K)$. By part 3 of Lemma 1.5.1, $h \circ g - id_K \in Z(K, K)$ and $g \circ h - id_{K'} \in Z(K', K')$. It suffices to check that $h \circ g$ and $g \circ h$ are invertible in $K(\underline{C})$; this follows from part 1 of Proposition 3.1.8.

2. By definition of WD , the triple $(g, WD(g))$ gives a morphism of weight decompositions.

Now suppose that $(h, i) = WD(g)$ i.e. $(h, i) \in C(\underline{C})(WD(X), WD(X'))$ and $(h, i) \equiv WD(g) \pmod{T(WD(X), WD(X'))}$. It follows that $i \circ f = f' \circ h$ (in the notation of (4)). Besides, there exist (h', i') that give a morphism of weight decompositions; $h - h' = s \circ f$ and $i - i' = f' \circ t$ for some $s, t \in \underline{C}(X^{w \geq 1}, X'^{w \leq 0})$. We obtain that $h \circ a = h' \circ a = a' \circ g$ and $b' \circ i = b' \circ i' = g[1] \circ b$.

Hence (g, h, i) give a morphism $T_0 \rightarrow T'_0$.

3. By definition, any $h \in K_{\mathfrak{w}}^{[0,1]}(\underline{C})(WD(X), WD(X'))$ comes from some commutative square

$$\begin{array}{ccc} X^{w \leq 0} & \xrightarrow{f_0} & X^{w \geq 1} \\ \downarrow & & \downarrow \\ X'^{w \leq 0} & \xrightarrow{f'_0} & X'^{w \geq 1} \end{array}$$

Extending this square to a morphisms of triangles $T_0 \rightarrow T'_0$ (i.e. of weight decompositions of X and X') immediately yields the result.

4. Since WD is a functor, T is an ideal.

We prove that $T^2 = 0$ similarly to the proof of I2 of Lemma 3.1.4.

Let $X, X', X'' \in \text{Obj } \underline{C}$, let $g \in T(X', X'') \subset \underline{C}(X, X')$, $h \in T(X', X'') \subset \underline{C}(X', X'')$.

We should check that $h \circ g = 0$ (in \underline{C}). We can choose any weight decompositions of X, X', X'' ; denote them by T, T', T'' (similarly to (4)).

Since $WD(g) = WD(h) = 0$, by assertion I2 we obtain that $(g, 0, 0)$ and $(h, 0, 0)$ give morphisms of weight decompositions. This means that $a' \circ g = a'' \circ h = g[1] \circ b = h[1] \circ b' = 0$. Hence g can be presented as $b'[-1] \circ c$ for some $c \in \underline{C}(X, X'^{w \geq 1}[-1])$. Then $h \circ g = (h[1] \circ b')[-1] \circ c = 0$.

5. By assertion I3 any isomorphism $WD(X) \rightarrow WD(X')$ is induced by some morphism $X \rightarrow X'$. Now by part II1 of Lemma 3.1.4, t is conservative (we apply assertion I4); this yields the result.

6. Immediate from part II2 of Lemma 3.1.4.

II Exactly the same reasoning as in part I1 will prove the assertion after we verify that morphisms in \underline{C} give well-defined morphisms of weight complexes (in $K_w(\underline{Hw})$).

A $g \in \underline{C}(X, X')$ can be extended to a morphism $Po(X) \rightarrow Po(X')$ (see part 1 of Remark 1.5.9); hence we also obtain some morphism $t(g) : t(X) \rightarrow t(X')$. It remains to verify that for $g = 0$ we have $t(g) \in Z(t(X), t(X'))$. We use the notation of part 2 of Proposition 1.5.6.

We study the possibilities for $g_i : X^i \rightarrow X'^i$ (that we choose to be compatible with (9), without considering (10)). By construction, g_i depends on the maps $r_k : X^{w \leq k} \rightarrow X'^{w \leq k}$ only for $k = i, i-1$. This dependence is linear. Moreover, any pair of (r_i, r_{i-1}) can be presented as $(0, r_{i-1}) + (r_i, 0)$. Indeed, for $g = 0$ any of r_k could be zero, whereas distinct r_k are 'independent' by part 1 of Proposition 1.5.6. Hence it suffices to prove that g_i can be presented as $(s_{i+1} \circ h_{iX} + h_{i-1, X'} \circ t_i)$ for some $s_{i+1} \in \underline{Hw}(X^{i+1}, X'^i)$, $t_i \in \underline{Hw}(X^i, X'^{i-1})$ in two cases: either r_i or r_{i-1} equals 0. (Recall that h denotes the boundary of a weight complex).

In the case $r_i = 0$ we can present $g_i[-1]$ as the second component of $WD(0 : X^{w \leq i}[-1] \rightarrow X'^{w \leq i}[-1])$. Hence g_i equals $c_{i-1, X'} \circ u_i$ for some $u_i \in \underline{C}(X^i, X'^{w \leq i-1})$ (by assertion I1). Note now that u_i can be factorized through X'^{i-1} (see Lemma 3.2.1).

In the case $r_{i-1} = 0$ we can present g_i as the first component of $WD(0 : X^{w \geq i} \rightarrow X'^{w \geq i})$. Hence g_i equals $v_{i+1} \circ x_{iX}[1]$ for some $v_{i+1} \in \underline{C}(X^{w \geq i+1}, X'^i)$. It remains to note that v_i can be factorized through X^{i+1} .

Combining the two cases, we obtain our claim. □

Remark 3.2.3. The functoriality of t implies that for any $X \in \text{Obj}\underline{C}$ any two choices for $t(X)$ are connected by a (canonical) isomorphism in $K_{\mathfrak{w}}(\underline{Hw})$. Then part II1 of Lemma 3.1.4 (combined with part I2 of the Lemma) implies that they are isomorphic (not necessarily canonically) in $K(\underline{Hw})$ i.e. they are homotopy equivalent (in $C(\underline{Hw})$).

WD and t commute in the following sense.

Proposition 3.2.4. *Let $X, X' \in \text{Obj}\underline{C}$, $g \in \underline{C}(X, X')$.*

1. Any choice of $(t(i), t(l))$ for $(i, l) = WD(g)$ comes from a truncation of $t(g)$ (here we fix some weight decompositions of X and X' and consider all compatible lifts of $t(g)$ to $\text{Mor}C(\underline{Hw})$).

2. Let some $(r', s') = (t(i'), t(l'))$ for some weight decomposition (i', l') of g , let $r + s : t(X) \rightarrow t(X')$ be homotopic to $r' + s'$ (here we consider sums of collections of arrows). Then $(r, s) = (t(i), t(l))$ for some (other) weight decomposition (i, l) of g .

Proof. 1. By the definition of $t(g)$ (see part II of Theorem 3.2.2) any choice of $(t(i), t(l))$ is a possible truncation of $t(h)$ over $C(\underline{Hw})$.

2. It suffices to prove the statement for $g = 0$. Suppose that (r, s) could be obtained from some $WD(0)$ via t . Note that (replacing r, s by equivalent morphisms if needed) we can assume that $r = r_0$, $s = s_1$ (i.e. they are concentrated in degrees 0, 1). Hence there exists some $l \in \underline{Hw}(X^1, X'^0)$ such that $r_0 = l \circ h_0$, $s_1 = h'_0 \circ l$.

Now it remains to note that the triple $(0, d'_0 \circ l \circ c_1, x'_0[1] \circ l \circ y_1)$ gives a weight decomposition of $0 : X \rightarrow X'$. This fact follows from the equalities $d'_0 \circ l \circ c_1 \circ a_0 = 0 = b'_0 \circ x'_0[1] \circ l \circ y_1$ (see (9) and (10)), whereas

$$f'_0 \circ d'_0 \circ l \circ c_1 = x'_0[1] \circ l \circ c_1 = x'_0[1] \circ l \circ y_1 \circ f_0.$$

□

3.3 Main properties of the weight complex

Now we prove the main properties of the weight complex functor.

Theorem 3.3.1. *[The weight complex theorem]*

I Exactness.

t is a weakly exact functor.

II Nilpotency.

$I(-, -) = \text{Ker}(\underline{C}(-, -) \rightarrow K_{\mathfrak{w}}(t(-), t(-)))$ defines an ideal in $\text{Mor}\underline{C}$. For any $i \leq j \in \mathbb{Z}$ the restriction $I^{[i, j]}$ of I to $\underline{C}^{[i, j]}$ satisfies $(I^{[i, j]})^{j-i+1} = 0$.

III Idempotents.

If $X \in \underline{C}^b$, $g \in \underline{C}(X, X)$, $t(g) = t(g \circ g)$, then $t(g)$ can be lifted to an idempotent $g' \in \underline{C}(X, X)$.

IV Filtration.

If $X \in \underline{C}^{w \leq i}$ (resp. $\underline{C}^{w \geq i}$) for some $i \in \mathbb{Z}$ then $t(X) \in K_{\mathbf{w}}(\underline{Hw})^{\leq i}$ (resp. $K_{\mathbf{w}}(\underline{Hw})^{\geq i}$) i.e. it is homotopy equivalent to a complex concentrated in degrees $\leq i$ (resp. $\geq i$).

If X is bounded from above (resp. from below) then the converse implications are valid also.

V Conservativity.

If w is non-degenerate, then the functor t is conservative on \underline{C}^+ and \underline{C}^- .

VI If $X, Y \in \underline{C}^{[0,1]}$ then $t(X) \cong t(Y)$ implies $X \cong Y$.

VII Let $X \in \underline{C}^{w \geq a}$ for some $a \in \mathbb{Z}$; consider the homomorphism $t_* : \underline{C}(X, X') \rightarrow K_{\mathbf{w}}(\underline{Hw})(t(X), t(X'))$. Then the following statements are valid.

1. If $X' \in \underline{C}^{w \leq a}$ then t_* is bijective.
2. If $X' \in \underline{C}^{w \leq a+1}$ then t_* is surjective.

Proof. I Let $C \xrightarrow{a} X \xrightarrow{f} X' \xrightarrow{b} C[1]$ be a distinguished triangle. We should prove that the triangle $t(C) \xrightarrow{t(a)} t(X) \xrightarrow{t(f)} t(X') \xrightarrow{t(b)} t(C)[1]$ is distinguished. It suffices to construct a triangle of morphisms

$$V : t(X'[-1]) \xrightarrow{m} t(C) \xrightarrow{n} t(X) \quad (17)$$

that splits componentwisely (in $C(\underline{Hw})$) such that m is some choice for $t(b)[-1]$ and n is some choice for $t(a)$. Indeed, it is a well known fact that any such V gives a distinguished triangle in $K(\underline{Hw})$. Hence any two sides of $t(V)$ can be lifted to two sides of a distinguished triangle in $K(\underline{Hw})$; so $t(V)$ is distinguished (see Definition 3.1.6).

In order to prove our claim we construct 'nice' weight decompositions of $C[i]$ for all $i \in \mathbb{Z}$. To this end we apply the method used in the proof of part III of Proposition 2.5.1 for all $k \in \mathbb{Z}$.

We apply Lemma 1.5.4 for all $i \in \mathbb{Z}$. By the lemma, the triangles $C[i] \rightarrow C_i \rightarrow C'_i$ obtained from (6) by shifting the last row are weight decompositions of $C[i]$ for all $i \in \mathbb{Z}$. Hence the first two columns of (6) can be completed to morphisms of the analogues of (9) for X' , $C[1]$, and $X[1]$ (for all $k \in \mathbb{Z}$).

Now, in order to 'connect' shifted weight decompositions we use exactly the same reasoning as in the proof of part III of Proposition 2.5.1. We obtain that the corresponding map of weight complexes splits componentwisely indeed.

II I is an ideal since t is an additive functor.

Obviously, it suffices to check that for $X \in \underline{C}^{[0,n]}$ the ideal $J = \{g \in \underline{C}(X, X) : t(g) = 0\}$ of the ring $\underline{C}(X, X)$ satisfies $J^{n+1} = 0$. We will prove

this fact by induction on n . In the case $n = 0$ we have $\underline{C}^{[0,n]} = \underline{Hw}$, hence $J = \{0\}$.

To make the inductive step we consider $g_0 \circ g_1 \circ \dots \circ g_n$, $g_i \in J$, let $r = (g_0 \circ g_1 \circ \dots \circ g_{n-1})[n-1]$, $s = g_n[n-1] \circ r$. By Proposition 3.2.4, we can choose a representative (h_i, l_i) of $WD(g_i[n-1])$ such that $t(h_i) = 0$. Then by the inductive assumption we have $WD(r) = (0, m)$ for some $m : X^n \rightarrow X^n$. Considering the morphism of triangles corresponding to $WD(r)$ we obtain that $r = b_{n-1}[-1] \circ q$ for some $q : X[n-1] \rightarrow X^n[-1]$. Next, since $t(g_n) = 0$, we can assume that $t(g_n[n-1]) = (u, 0)$ for some u (by Proposition 3.2.4). Hence $g_n[n-1] = v \circ a_{n-1}$ for some $v \in \underline{C}(X^{w \leq n-1}, X[n-1])$ and we obtain $s = v \circ (a_{n-1} \circ b_{n-1}[-1]) \circ q = 0$. The assertion is proved.

III Follows from assertion II by a standard reasoning; see Remark 3.1.5.

IV By part 2 of Proposition 2.2.2: if $X \in \underline{C}^{w \leq i}$ (resp. $X \in \underline{C}^{w \geq i}$) then choosing $X^{w \geq i+1} = 0$ (resp. $X^{w \leq i-1} = 0$) we obtain that the corresponding choice of $t(X)$ is concentrated in degrees $\leq i$ (resp. $\geq i$). Now note that all choices of $t(X)$ are homotopy equivalent by part 1 of Proposition 3.1.8.

Conversely, let w be non-degenerate, let $t(X) \in K_{\mathfrak{w}}(\underline{Hw})^{w \leq i}$. We can assume that $i = 0$; let $X \in \underline{C}^{w \leq n}$ (for some $n \geq 0$). Then $t(id_X)$ is equivalent to a morphism whose non-zero components are in degrees ≤ 0 . Hence Proposition 3.2.4 implies that for $WD(id_X) = (l, m)$ we can assume that $t(m) = 0$.

Then by assertion II we have $WD(id_X^n) = (l^n, 0)$. Considering the distinguished triangle corresponding to $WD(id_X^n)$ we obtain that $id_X = id_X^n$ can be factorized through $X^{w \leq 0}$. Hence X is a retract of $X^{w \leq 0}$; since $\underline{C}^{w \leq 0}$ is Karoubi-closed in \underline{C} we obtain that $X \in \underline{C}^{w \leq 0}$.

The case $t(X) \in K_{\mathfrak{w}}(\underline{Hw})^{w \geq i}$ is considered similarly.

V Since t is weakly exact (see Definition 3.1.6), it suffices to check that $t(X) = 0$ implies $X = 0$. This is immediate from assertion IV.

VI Immediate from part I5 of Theorem 3.2.2.

VII We can assume that $a = 0$.

1. The proof is just a repetitive application of axioms (of weight structures).

Note first that t_* is bijective for $X, X' \in \underline{C}^{w=0}$. Next, for $X \in \underline{C}^{w=0}$ and any X' we consider the distinguished triangle $X'^{w \leq -1} \rightarrow X'^0 \rightarrow X' \rightarrow X'^{w \leq -1}[1]$. Then orthogonality yields that any $h : \underline{C}(X, X'^0)$ gives a morphism $X \rightarrow X'$; hence t is surjective in this case. We also can apply this

statement for $X'' = X'^{w \leq -1}$. Hence considering the diagram

$$\begin{array}{ccccccc}
(\underline{C}(X, X'^{w \leq -1})) & \longrightarrow & \underline{C}(X, X'^0) & \longrightarrow & \underline{C}(X, X') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_{\mathfrak{w}}(\underline{Hw})(t(X), t(X'^{w \leq -1})) & \longrightarrow & K_{\mathfrak{w}}(\underline{Hw})(t(X), t(X'^0)) & \longrightarrow & K_{\mathfrak{w}}(\underline{Hw})(t(X), t(X')) & \longrightarrow & 0
\end{array}$$

induced by t we obtain that t_* is bijective in this case.

Now considering the distinguished triangle $X^{w \geq 1}[-1] \rightarrow X \rightarrow X^0 \rightarrow X^{w \geq 1}$ and applying the dual argument one can easily obtain the claim.

2. Let $h \in K_{\mathfrak{w}}(\underline{Hw})(t(X), t(X'))$. By definition, we can 'cut' h to obtain a commutative diagram

$$\begin{array}{ccc}
t(X^0) & \xrightarrow{t(f_0)} & t(X^{w \geq 1}) \\
\downarrow & & \downarrow \\
t(X')^{w \leq 0} & \xrightarrow{t(f'_0)} & t(X'^1)
\end{array}$$

By assertion VIII1, this diagram corresponds to some homomorphism $WD(X) \rightarrow WD(X')$. It remains to apply part I3 of Theorem 3.2.2. \square

Remark 3.3.2. 1. By assertions IV and V, t is always conservative and strictly respects weight filtration on \underline{C}^b .

2. In fact, our restrictions on a distinguished triangle in $K_{\mathfrak{w}}(A)$ (see Definition 3.1.6) are rather weak. Our definition is similar to the notion of an *exact triangle* in Definition 0.3 of [Vak01]. Since exact triangles are not distinguished in general (see loc.cit.), part I of our theorem does not imply that for a distinguished triangle $C \rightarrow X \rightarrow X'$ the triangle $t(C) \rightarrow t(X) \rightarrow t(X')$ comes from some distinguished triangle in $K(\underline{Hw})$.

We will not prove the latter fact in detail, since we will only need it in Remark 3.3.4 below. Yet the proof is rather easy. In the proof of part I of our theorem it suffices to check that some choice of $t(f)$ in $K(\underline{Hw})$ yields the third side of the distinguished triangle in question. Using (obvious) functoriality properties of the construction in the proof, one can reduce the latter claim to the case $X, X' \in \underline{C}^{w=0}$. Certainly, the statement is obvious in this case.

Probably t could be lifted to a certain 'strong' weight complex functor.

Conjecture 3.3.3. t could be lifted to an exact functor $t^{st} : \underline{C} \rightarrow K(\underline{Hw})$.

Remark 3.3.4. 1. Let \underline{Hw} be (fully) embedded into the subcategory $B = \text{Proj } A$ of projective objects of an abelian category A (the most reasonable choice for A is \underline{Hw}'_* ; cf. Lemma 5.4.3 below). Then we have full embeddings

$K(\underline{Hw}) \subset K(B) \subset D(A)$. Suppose now that A is of projective dimension 1. Then any complex over A is quasi-isomorphic to a complex with zero differentials; hence it can be presented in $D(A)$ as a direct sum of some monomorphisms in B (i.e. of complexes of the form $\dots 0 \rightarrow X \xrightarrow{i} Y \rightarrow 0 \rightarrow \dots$ placed in pairwise distinct dimensions). We check that $K_{\mathfrak{w}}(B) = K(B)$. Note that it suffices to prove the corresponding fact for $K^b(-)$. Therefore it suffices to check that $K_{\mathfrak{w}}(B)(X, Y) = K(B)(X, Y)$ for X, Y being monomorphisms (as two-term complexes); let $X = X^{-1} \hookrightarrow X^0$. If $Y \in C^{[-1, 0]}(B)$ then $K(B)(X, Y) = A(H^0(X), H^0(Y)) = K_{\mathfrak{w}}(B)(X, Y)$ (see part 2 of Remark 3.1.7). If $Y \in C^{[-2, -1]}(B)$ then the equality $K(B)(X, Y) = K_{\mathfrak{w}}(B)(X, Y)$ is obvious (cf. part VII of Theorem 3.3.1). For Y placed in all other positions we have $K(B)(X, Y) = \{0\} = K_{\mathfrak{w}}(B)(X, Y)$.

We conclude that $K_{\mathfrak{w}}(\underline{Hw}) = K(\underline{Hw})$. Therefore part 2 of Remark 3.3.2 implies that t is exact (as a functor of triangulated categories).

In particular, this reasoning can be applied if $\underline{Hw} = Ab_{fin.fr}$ or $\underline{Hw} = Ab_{fr}$. Hence this is the case for all categories of spectra considered in §4.6 below.

2. In §6.3 below we will also verify the conjecture in the case when \underline{C} has a *differential graded enhancement*.

3. Prof. A. Beilinson has kindly communicated to the author a proof of the conjecture in the case when \underline{C} has a *filtered triangulated enhancement*; see §8.4 below. Probably, a filtered triangulated enhancement exists for any 'reasonable' triangulated category.

4 Relating weight structures and t -structures: adjacent structures and adjoint functors

In this section (especially in §4.4) we prove that weight structures are closely related to t -structures.

In §4.1 we recall the definition of a t -structure in a triangulated \underline{C} . In §4.2 we recall the (standard) construction of countable homotopy colimits in triangulated categories and study its properties.

In §4.3 we show that in many cases a weight structure can be described by specifying a *negative* $H(' \approx' \underline{Hw}) \subset \underline{C}$. In particular, this is the case for the category of finite spectra ($\subset SH$).

In §4.4 we define the notion of (left or right) *adjacent* weight and t -structures for \underline{C} ; their hearts are dual in a very interesting sense (see Theorem 4.4.2). It turns out that the truncations of an object Y with respect to a t -structure that is adjacent to w represent exactly the virtual t -truncations

of the functor $\underline{C}(-, Y)$ (with respect to w , see Remark 2.5.2). Hence spectral sequences arising from adjacent weight and t -structures are closely related. Lastly, a functor of triangulated categories is t -exact (with respect to some t -structures) whenever its (left) adjoint is *weight-exact* with respect to weight structures that are (left) adjacent to these t -structures.

In §4.5 we study the conditions for adjacent weight and t -structures to exist. We only consider in detail the cases which are relevant for our main examples (motives and spectra); other possibilities are mentioned in Remark 4.5.3. Note also that the *weight resolution* construction used in the proof of Theorem 4.5.2 allows to construct Eilenberg-MacLane spectra in SH .

In 4.6 we apply the results of this section to the study of SH . In particular, we construct a *spherical* weight structure for it; it is adjacent to the Postnikov t -structure.

In §7.1 below we will apply our results to DM_-^{eff} (the category of motivic complexes of Voevodsky).

4.1 t -structures: reminder

To fix the notation we recall the definition of a t -structure.

Definition 4.1.1. A pair of subclasses $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0} \subset \text{Obj} \underline{C}$ will be said to define a t -structure t if they satisfy the following conditions:

(i) $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0}$ are strict i.e. contain all objects of \underline{C} isomorphic to their elements.

(ii) $\underline{C}^{t \geq 0} \subset \underline{C}^{t \geq 0}[1], \underline{C}^{t \leq 0}[1] \subset \underline{C}^{t \leq 0}$.

(iii) **Orthogonality.** $\underline{C}^{t \leq 0}[1] \perp \underline{C}^{t \geq 0}$.

(iv) **t -decompositions.**

For any $X \in \text{Obj} \underline{C}$ there exists a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow A[1] \quad (18)$$

such that $A \in \underline{C}^{t \leq 0}, B \in \underline{C}^{t \geq 0}[-1]$.

Non-degenerate and bounded (above, below, or both) t -structures could be defined similarly to Definition 1.2.1.

We will need some more notation for t -structures.

Definition 4.1.2. 1. A category \underline{Ht} whose objects are $\underline{C}^{t=0} = \underline{C}^{t \geq 0} \cap \underline{C}^{t \leq 0}$, $\underline{Ht}(X, Y) = \underline{C}(X, Y)$ for $X, Y \in \underline{C}^{t=0}$, will be called the *heart* of t . Recall (cf. Theorem 1.3.6 of [BBD82]) that \underline{Ht} is always abelian; short exact sequences in \underline{Ht} come from distinguished triangles in \underline{C} .

2. $\underline{C}^{t \geq l}$ (resp. $\underline{C}^{t \leq l}$) will denote $\underline{C}^{t \geq 0}[-l]$ (resp. $\underline{C}^{t \leq 0}[-l]$).

Remark 4.1.3. 1. Recall (cf. Lemma IV.4.5 in [GeM03]) that (18) defines additive functors $\underline{C} \rightarrow \underline{C}^{t \leq 0} : X \rightarrow A$ and $C \rightarrow \underline{C}^{t \geq 1} : X \rightarrow B$. We will denote A, B by $X^{t \leq 0}$ and $X^{t \geq 1}[-1]$, respectively. (18) will be called the *t-decomposition* of X .

More generally, the t -components of $X[i]$ will be denoted by $X^{t \leq i} \in \underline{C}^{t \leq 0}$ and $X^{t \geq i+1}[-1] \in \underline{C}^{t \geq 1}$, respectively.

2. The functor $X \rightarrow X^{t \leq 0}$ is right adjoint to the inclusion $\underline{C}^{t \leq 0} \rightarrow \underline{C}$. It follows that this functor commutes with all those coproducts that exist in \underline{C} . Besides, if $\coprod X_i$, $\coprod X_i^{t \leq 0}$, and $\coprod X_i^{t \geq 1}$ exist in \underline{C} , then the distinguished triangle $\coprod X_i^{t \leq 0} \rightarrow \coprod X_i \rightarrow \coprod X_i^{t \geq 1}[-1]$ yields that $(\coprod X_i)^{t \geq 1} = \coprod (X_i^{t \geq 1})$.

We denote by H_t^0 the zeroth cohomology functor corresponding to t (cf. part 10 of §IV.4 of [GeM03]); i.e. $H_t^0(X)$ is defined similarly to $X^{[0,1]}$ in part 1 of Proposition 1.5.6. Shifting the t -decomposition of $X^{t \leq 0}[-1]$ by $[1]$ we obtain a canonical and functorial (with respect to X) distinguished triangle $X^{t \leq -1}[1] \rightarrow X^{t \leq 0} \rightarrow H_t^0(X)$.

Lastly, $\tau_{\leq i}X$ will denote $X^{t \leq i}[-i]$; $\tau_{\geq i}X = X^{t \geq i}[-i]$; $H_t^i = H_t^0(X[i])$.

4.2 Countable homotopy colimits in triangulated categories: the construction and properties

The triangulated construction of countable (filtered) homotopy colimits is fairly standard, cf. Definition 1.6.4 of [Nee01].

Definition 4.2.1. Suppose that we have a sequence of objects Y_i (starting from some $j \in \mathbb{Z}$) and maps $\phi_i : Y_i \rightarrow Y_{i+1}$. Let there exist $D = \coprod Y_i$ in \underline{C} . We consider the map $d : \oplus id_{Y_i} \oplus \oplus (-\phi_i) : D \rightarrow D$ (we can define it since its i -th component is can be easily factorized as a composition $Y_i \rightarrow Y_i \oplus Y_{i+1} \rightarrow D$). Denote a cone of d as Y . We will write $Y = \varinjlim Y_i$ and call Y the *homotopy colimit* of Y_i ; we will not consider any other homotopy colimits in this paper.

We will say that the colimit exists (in \underline{C}) if the coproduct D exists.

Remark 4.2.2. 1. By Lemma 1.7.1 of [Nee01] the homotopy colimit of Y_{i_j} is the same for any subsequence of Y_i . In particular, we can discard any (finite) number of first terms in Y_i .

2. By Lemma 1.6.6 of [Nee01] the homotopy colimit of $X \xrightarrow{id_X} X \xrightarrow{id_X} X \xrightarrow{id_X} \dots$ is X . Hence we obtain that $\varinjlim X_i \cong X$ if for $i \gg 0$ all ϕ_i are isomorphisms and $X_i \cong X$.

3. The construction of $\varinjlim Y_i$ easily yields: if countable coproducts exist in $\underline{C}^{w \leq 0}$ then $\underline{C}^{w \leq 0}$ is closed (in \underline{C}) with respect to (countable) homotopy

colimits. Indeed, we have $D \in \underline{C}^{w \leq 0}$; hence it suffices to recall that $\underline{C}^{w \leq 0}$ is extension-stable (see part 3 of Proposition 1.3.3). On the other hand, it is easy to construct a counterexample to the similar statement for $\underline{C}^{w \geq 0}$ (though countable colimits of objects of $\underline{C}^{w \geq 0}$ always belong to $\underline{C}^{w \geq -1}$). To settle this problem below we will describe a 'clever' method for passing to the colimit in $\underline{C}^{w \geq 0}$.

We study the behaviour of colimits under (co)representable functors.

Lemma 4.2.3. 1. For any $C \in \text{Obj } \underline{C}$ we have a natural surjection $\underline{C}(Y, C) \rightarrow \varprojlim \underline{C}(Y_i, C)$.

2. This epimorphism is bijective if all $\phi_i[1]^* : \underline{C}(Y_{i+1}[1], C) \rightarrow \underline{C}(Y_i[1], C)$ are surjective for all $i \gg 0$.

3. If C is compact then $\underline{C}(C, Y) = \varinjlim \underline{C}(C, Y_i)$.

Proof. 1. For any C we have $\underline{C}(D, C) = \prod \underline{C}(Y_i, C)$.

This yields a long exact sequence

$$\cdots \rightarrow \underline{C}(D[1], C) \xrightarrow{a[1]^*} \underline{C}(D[1], C) \rightarrow \underline{C}(Y, C) \rightarrow \underline{C}(D, C) \xrightarrow{a^*} \underline{C}(D, C) \rightarrow \cdots$$

It is easily seen that the kernel of a^* equals

$$\{(s_i) : s_i \in \underline{C}(Y_i, C), s_{i+1} = s_i \circ \phi_i\} = \varprojlim \underline{C}(Y_i, C);$$

this yields the result.

2. By part 1 of Remark 4.2.2, we can assume that the homomorphisms $\phi[1]^*$ are surjective for all i . In this case $a[1]^*$ is easily seen to be surjective; this yields the result.

3. Similarly to the proof of part 1, we consider the long exact sequence

$$\cdots \rightarrow \underline{C}(C, D) \xrightarrow{a_*} \underline{C}(C, D) \rightarrow \underline{C}(C, Y) \rightarrow \underline{C}(C, D[1]) \xrightarrow{a[1]^*} \underline{C}(C, D[1]) \rightarrow \cdots$$

Since C is compact, we have $\underline{C}(C, D) = \bigoplus \underline{C}(C, Y_i)$. Then it is easily seen that $a[1]_*$ is surjective, whereas the cokernel of a_* is $\varinjlim \underline{C}(C, Y_i)$. See also Lemma 2.8 of [Nee96].

□

Now we describe a 'clever' method for passing to the colimit in $\underline{C}^{w \geq 0}$. Since we will use it to prove that a certain candidate for being a weight structure is a weight structure indeed, we will describe it in a (somewhat) more general setting than those of weight structures.

Suppose that we have a full extension-stable (see Definition 1.3.1) subcategory $D \subset \underline{C}$. Define a full subcategory $E \subset \underline{C}$ by $\text{Obj } E = {}^\perp(D[1])$. Note that E is also extension-stable, whereas the pair (D, E) satisfies the conditions of Remark 1.5.5.

Lemma 4.2.4. *Let $\phi_i : Y_i \rightarrow Y_{i+1}$ be a sequence of \underline{C} -morphisms; denote $\text{Cone } \phi_i$ by C_i ; let the first of Y_i be Y_l . Suppose that Y_l and all C_i have 'weight decompositions with respect to D, E ' i.e. that there exist distinguished triangles $Y_l \rightarrow D_l \rightarrow H_l$ and $C_i \rightarrow F_i \rightarrow G_i$ with $D_l, F_i \in \text{Obj } D$ and $H_l, G_i \in \text{Obj } E$. Suppose also that for any possible choice of 'weight decompositions' of Y_i ($Y_i \rightarrow D_i \rightarrow E_i$ with $D_i \in \text{Obj } D$ and $E_i \in \text{Obj } E$) the coproduct $\coprod E_i$ exists. Then there exists a choice of E_i and of the morphisms $\phi'_i : E_i \rightarrow E_{i+1}$ compatible with ϕ_i such that $\varinjlim E_i \in \text{Obj } E$ (note that the colimit exists!).*

Proof. We fix weight decompositions for all C_i and for Y_l .

Next we fix ϕ'_i and the weight decompositions of Y_{i+1} starting from $i = l$ inductively.

Suppose that we have fixed some weight decomposition of Y_i . By Remark 1.5.5 we can construct E_{i+1} and ϕ'_i that fit into a distinguished triangle $E_i \xrightarrow{\phi'_i} E_{i+1} \rightarrow G_i$, whereas $E_{i+1}[-1] \rightarrow Y_{i+1}$ yields a weight decomposition of Y_{i+1} .

Now we check that passing to the limit of E_i this way we obtain an object of E . Let Z be the limit of E_i . We should check that $Z \perp D[1]$. By part 2 of Lemma 4.2.3, to this end it suffices to check that all $\phi'_i[1]^* : \underline{C}(E_{i+1}, C) \rightarrow \underline{C}(E_i^{w \geq 1}[1], C)$ are surjective. Indeed, then we will have $\underline{C}(Z, C) = \varprojlim \underline{C}(E_i, C) = \{0\}$. Lastly, the surjectivity is immediate from the long exact sequences (for all i)

$$\cdots \rightarrow \underline{C}(E_{i+1}[1], C) \rightarrow \underline{C}(E_i[1], C) \rightarrow \underline{C}(G_i, C)(= \{0\}) \rightarrow \cdots$$

□

Remark 4.2.5. Note that t -structures do not have a similar property (since there is no flexibility for t -decompositions).

Lastly we prove that t -truncations 'approximate' objects. We will prove this statement in the form that is relevant for §7.1; certainly, some other versions of it are valid for similar reasons.

Lemma 4.2.6. *Let t be a non-degenerate t -structure. Suppose that all countable coproducts exist in $\underline{C}^{t \leq 0}$; suppose also that (countable) coproducts respect t -decompositions of objects of $\cup_{i \in \mathbb{Z}} \underline{C}^{t \leq i}$.*

Let all $Y_i \in \underline{C}^{t \leq l}$ for some $l \in \mathbb{Z}$, let $\phi_i : Y_i \rightarrow Y_{i+1}$ be a sequence of \underline{C} -morphisms. Suppose that there exists such an Y that for any $j \in \mathbb{Z}$ and all $i \geq j$ we have $Y_i^{t \geq -j} \cong Y^{t \geq -j}$ and these isomorphisms commute with $\phi_{i} : Y_i^{t \geq -j} \rightarrow Y_{i+1}^{t \geq -j}$. Then $\varinjlim Y_i$ exists and $\cong Y$.*

Proof. Since countable coproducts exist in $\underline{C}^{t \leq 0}$, they also exist in $\underline{C}^{t \leq l}$. This implies the existence of $\varinjlim Y_i$. We denote $\varinjlim Y_i$ by Z .

We obviously have $Y \in \underline{C}^{t \leq l}$. Then the definition of \varinjlim easily yields (at least, one) morphism $\varinjlim(Y \xrightarrow{id_Y} Y \xrightarrow{id_Y} Y \xrightarrow{id_Y} \dots) = Y \rightarrow \varinjlim Y_i = Z$; to this end one should apply Proposition 1.4.2.

Since t is non-degenerate, it suffices to prove that $H_t^k(Y) \cong H_t^k(Z)$ for any $k \in \mathbb{Z}$. We fix k .

Since countable coproducts respect t -cohomology in question, we have a long exact sequence (in $\underline{H}t$): $\dots \rightarrow \coprod H_t^k(Y_i) \rightarrow H_t^k(Z) \rightarrow \coprod H_t^{k+1}(Y_i) \rightarrow \dots$. We can also obtain a similar long exact sequence for $H_t^k(Z)$ (with Y_i replaced by Y) if we present it as $\varinjlim(Y \xrightarrow{id_Y} Y \xrightarrow{id_Y} Y \xrightarrow{id_Y} \dots)$. Now, by part 1 of Remark 4.2.2 we can assume that all $H_t^k(Y_i) \cong H_t^k(Y)$ and $H_t^{k+1}(Y_i) \cong H_t^{k+1}(Y)$. This concludes the proof. \square

4.3 Recovering w from its heart

In many cases instead of describing $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ it is easier to specify only $\underline{C}^{w=0}$. We describe some conditions that ensure that w can be recovered from $\underline{H}w$. We will need the following definitions.

Definition 4.3.1. Let H be a full additive subcategory of \underline{C} .

1. We will say that H is *negative* if $Obj H \perp (\cup_{i>0} Obj(H[i]))$.
2. We will say that $H' \subset \underline{C}$ is the *Karoubi-closure* of H if the objects of H' are exactly all retracts of objects of H (in \underline{C}).
3. We define the *small envelope* of an additive category A as a category A' whose objects are (X, p) for $X \in Obj A$ and $p \in A(X, X)$ such that $p^2 = p$ and there exist $Y \in Obj A$ and $q \in A(X, Y)$, $s \in A(Y, X)$ satisfying $sq = 1 - p$, $qs = id_Y$. We define

$$A'((X, p), (X', p')) = \{f \in A(X, X') : p'f = fp = f\}. \quad (19)$$

The small envelope of A is (naturally) a full subcategory of the idempotent completion of A (cf. §5.1 below). One should think of A' as of the category of $X \oplus Y$ for $X, Y \in Obj A$, Y is a retract X . Here $X \oplus Y$ is a certain 'complement' of Y to X .

It can be easily checked that the small envelope of an additive category is additive; $X \rightarrow (X, id_X)$ gives a full embedding $A \rightarrow A'$.

Theorem 4.3.2. *Let A be a full additive subcategory of some triangulated \underline{C} . Then the embedding $A \rightarrow \underline{C}$ can be extended to a full embedding of the small envelope of A into \underline{C} .*

II Let H be negative and generate \underline{C} . Then the following statements are valid.

1. There exists a unique weight structure w for \underline{C} such that $H \subset \underline{H}w$. Moreover, it is bounded.

2. $\underline{H}w$ equals the Karoubi-closure of H in \underline{C} . The latter is equivalent to the small envelope of H .

III Let H be negative and weakly generate \underline{C} ; suppose that for any $X \in \text{Obj}\underline{C}$ there exists a $j \in \mathbb{Z}$ such that

$$\text{Obj}H \perp \{X[i], i > j\}. \quad (20)$$

Let $H' \subset H$ be additive. Suppose that one of the following conditions is fulfilled.

(i) There exists an infinite cardinality c such that any coproduct of $< c$ objects of H exists and belongs to H , whereas $\text{Card } H' < c$. For any $X \in \text{Obj}\underline{C}$ and any $Y \in \text{Obj}H'$ the group $\underline{C}(Y, X)$ considered as a $\underline{C}(Y, Y)$ -module can be generated by $< c$ elements. Any object of H can be presented as $\coprod_{i \in I} C_i$ for $C_i \in \text{Obj}H'$, $\text{Card } I < c$. For any $I : \text{Card}(I) < c$, $Y \in \text{Obj}H'$, $j \in \mathbb{Z}$, and $X_i \in \text{Obj}H$, $i \in I$, we have

$$\underline{C}(Y, \coprod_{j \in I} X_j) = \bigoplus \underline{C}(Y, X_j) \quad (21)$$

or

(ii) Arbitrary coproducts exist in H ; all objects of H' are compact; $\text{Obj}H'$ is a set; any object of H can be presented as $\coprod_{i \in I} C_i$ for $C_i \in \text{Obj}H'$ and some set I .

Then there exists a weight structure w for \underline{C} such that $\underline{H}w$ is the Karoubi-closure of H in \underline{C} . Moreover, w is non-degenerate and bounded above. In case (ii) it admits negative coproducts, in case (i) it admits negative coproducts of $< c$ objects.

IV Suppose that all conditions of part III ((i) or (ii)) except (20) are fulfilled. Denote the set of objects of \underline{C} satisfying (20) for some $j \in \mathbb{Z}$ by \underline{C}^- ; denote the class of objects of \underline{C} satisfying (20) for a fixed $j \in \mathbb{Z}$ by $\underline{C}^{w \leq j}$. Then the category \underline{C}^- is triangulated and satisfies all conditions of part III (we will identify the class \underline{C}^- with the corresponding full subcategory of \underline{C}).

Proof. I We map (X, p) to (any choice of) $\text{Cone}(q)$; we denote this object by Z .

Now we define the embedding on morphisms. We note that in A the map q is a projection of X onto Y . Hence in A' we have $X \cong (X, p) \oplus Y$, the isomorphism is given by (p, q) . Since q has a section in \underline{C} , we have a

distinguished triangle $Z \rightarrow X \xrightarrow{q} Y \xrightarrow{0} Z[1]$ i.e. we also have a similar decomposition of X in \underline{C} . It is easily seen that $\underline{C}(Z, Z')$ is given exactly by the formula (19) if we assume that Z is a subobject of X i.e. if we fix the splitting of the projection $X \rightarrow Z$. Hence if we fix the embedding $Z \rightarrow X$ for each (X, p) then (all possible choices) of objects $\text{Cone}(q)$ would give a subcategory that is equivalent to the small envelope of A ; it is obviously additive.

Alternatively, the statement can be easily deduced from the functoriality of the idempotent completion procedure (proved in [BaS01]; see §5.1 below).

II 1. We define $\underline{C}^{w \geq 0'}$ as the smallest extension-stable (see Definition 1.3.1) subclass of $\text{Obj} \underline{C}$ that contains $\text{Obj} H[i]$ for $i \leq 0$; for $\underline{C}^{w \leq 0'}$ we take a similar 'closure' of the set $\cup \text{Obj} H[i]$ for $i \geq 0$.

Obviously, $\underline{C}^{w \geq 0'}$ and $\underline{C}^{w \leq 0'}$ satisfy property (ii) of Definition 1.1.1; we define $\underline{C}^{w \geq i'}$ and $\underline{C}^{w \leq i'}$ for $i \in \mathbb{Z}$ in the usual way.

If we have a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ with $\underline{C}(X, A) = \underline{C}(Z, A) = \{0\}$ for some $X, Y, Z, A \in \text{Obj} \underline{C}$, then $\underline{C}(Y, A) = \{0\}$; the same statement is valid for a functor of the type $\underline{C}(B, -)$. Hence from the fact that $H[i] \perp H[j]$ for all $i < 0 \leq j$, we obtain (by induction) that $\underline{C}^{w \geq 1'} \perp \underline{C}^{w \leq 0'}$.

Now we verify that any $X \in \text{Obj} \underline{C}$ has a 'weight decomposition' (with respect to $\underline{C}^{w \geq 0'}$ and $\underline{C}^{w \leq 0'}$). We prove this by induction on the 'complexity' of X i.e. on the number of distinguished triangles that we have to consider to obtain X from objects of $H[i]$, $i \in \mathbb{Z}$ (by considering cones of morphisms).

For X of 'complexity' zero (i.e. for $X \in \text{Obj} H[i]$) we can take a trivial weight decomposition i.e. put $X^{w \leq 0}$ equal to X for $i \geq 0$ and to 0 otherwise; $X^{w \geq 0}$ will be 0 and X , respectively.

Suppose now that $X \cong \text{Cone}(Y \xrightarrow{d} Z)$ for Y, Z of 'complexity' less than that of X . By the inductive assumption there exist 'weight decompositions' of $Y[1]$ and Z i.e. distinguished triangles $Y[1] \xrightarrow{a} A \rightarrow B$ and $Z \xrightarrow{a'[-1]} A'[-1] \rightarrow B'[-1]$ for $A \in \underline{C}^{w \leq 0'}$, $A' \in \underline{C}^{w \leq -1'}$, $B \in \underline{C}^{w \geq 1'}$, $B' \in \underline{C}^{w \geq 0'}$. We apply Remark 1.5.5 for $D = \underline{C}^{w \leq 0'}$ and $E = \underline{C}^{w \leq 0'}$. It yields a 'weight decomposition' of X .

Now we take for $\underline{C}^{w \geq 0}$ and $\underline{C}^{w \leq 0}$ the Karoubi-closures of $\underline{C}^{w \geq 0'}$ and $\underline{C}^{w \leq 0'}$, respectively. By part 3 of Lemma 1.3.7, they satisfy the orthogonality axiom of weight structures. Hence they define a weight structure w for \underline{C} .

Now, since any object of \underline{C} can be obtained by a finite sequence of considerations of cones of morphisms from objects of \underline{Hw} , we obtain that w is bounded.

It remains to check that w is the only weight structure such that $H \subset \underline{Hw}$. By part 3 of Proposition 1.3.3, for any weight structure u satisfying $H \subset \underline{Hu}$ we have $\underline{C}^{w \geq 0'} \subset \underline{C}^{u \geq 0}$ and $\underline{C}^{w \leq 0'} \subset \underline{C}^{u \leq 0}$. Since $\underline{C}^{u \geq 0}$ and $\underline{C}^{u \leq 0}$ are Karoubi-

closed in \underline{C} , we also have $\underline{C}^{w \geq 0} \subset \underline{C}^{u \geq 0}$ and $\underline{C}^{w \leq 0} \subset \underline{C}^{u \leq 0}$. Now Lemma 1.3.8 implies our claim immediately.

2. By assertion I, \underline{C} contains the small envelope of H . To check that this envelope is actually contained in \underline{Hw} it suffices to note that the object $X \oplus Y$ can be presented both as a cone of the 'embedding' $Y \rightarrow X$ and of the projection $X \rightarrow Y$.

Now we verify the inverse inclusion; let $X \in \underline{C}^{w=0}$.

We apply the weight complex functor t . We obtain that $t(X) = X$ is a $K_w^b(\underline{Hw})$ -retract of two objects $A, B \in \text{Obj} K_w^b(\underline{Hw})$: $A \in C^{b, \geq 0}(H)$ and $B \in C^{b, \leq 0}(H)$ (here we use the descriptions of $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ given above, and the weak exactness of t). Next, applying Lemma 3.1.4 we obtain that the same is true in $K^b(\underline{Hw})$. Since t is also conservative, we can replace \underline{C} by $K^b(\underline{Hw})$ (with the same H); the assertion in this case could be verified 'by hand'.

III Again, for $\underline{C}^{w \geq 0}$ we take the smallest Karoubi-closed extension-stable subset of $\text{Obj} \underline{C}$ that contains $H[i]$ for $i \leq 0$.

We take $\underline{C}^{w \leq 0} = (\cup_{i < 0} \text{Obj} H[i])^\perp$. We prove orthogonality using induction on the 'complexity' of $Y \in \underline{C}^{w \geq 0}$ (as we also did in the proof of assertion III1). We have $Y \perp \underline{C}^{w \leq -1}$ for any $Y \in \underline{C}^{w \geq 0}$ of 'complexity zero' (i.e. $Y \in H$). Now using the fact that all (co)representable functors are homological on \underline{C} , we obtain that the same is true for objects of $\underline{C}^{w \geq 0} \cap \langle H \rangle$ of arbitrary complexity. Obviously, the same is true for their retracts i.e. for the whole $\underline{C}^{w \geq 0}$.

$\underline{C}^{w \leq 0}$ is Karoubi-closed and extension-stable by part 1 of Lemma 1.3.7; $\underline{C}^{w \geq 0}$ also fulfills these properties.

To prove that w is a weight structure, it remains to prove the existence of weight decompositions (see part 2 of Lemma 1.3.7). We will construct $X^{w \leq 0}$ and $X^{w \geq 1}$ for a fixed $X \in \text{Obj} \underline{C}$ explicitly. The construction could be called the *weight resolution*, cf. the proof of Proposition 4.5.2 below and Proposition 7.1.2 of [HPS97].

First we treat case (i). For each object Y of H' any $Z \in \text{Obj} \underline{C}$ we choose a set of $f_i(Y, Z) \in \underline{C}(Y, Z)$ of cardinality $< c$ so that $f_i(Y, Z)$ are $\underline{C}(Y, Y)$ -generators of $\underline{C}(Y, Z)$. Let $j \in \mathbb{Z}$ satisfy (20).

Now we construct a certain sequence of X_k for $k \leq j$ starting from $X_j = X$. For $k = j$ we take $P_j = \coprod_{Y \in \text{Obj} H', f_i(Y, X_j[j])} Y$. Note that the number of summands is $< c$, hence the sum exists and belongs to $\text{Obj} H$. Then we have a morphism $f_j : P_j \rightarrow X_j[j]$ given by $\prod f_i(Y, X[j])$. Let $X_{j-1}[j]$ denote a cone of f_j . Repeating the construction for X_{j-1} instead of X_j and with $k = j - 1$ we get an object $P_{j-1} \in \text{Obj} H'$, $f_{j-1} : P_{j-1} \rightarrow X_{j-1}[j - 1]$; we denote a cone of f_1 by $X_{j-2}[j - 1]$. Proceeding, we get an infinite sequence of (P_i, f_i, X_i) . Note that we have $P_i \in \underline{C}^{w \geq 0}$.

We denote the maps $X_i \rightarrow X_{i-1}$ given by the construction by g_i , $h_i = g_j \circ \dots \circ g_{i+2} \circ g_{i+1} : X \rightarrow X_i$. We denote a cone of h_i by $Y_i[-1]$; the map $Y \rightarrow X_i$ given by the corresponding distinguished triangle by r_i . Then part 3 of Remark 1.5.9 yields: $Y_i[i] \in \underline{C}^{rw \geq 0}$ for all $i \leq j$ (see the definition of $\underline{C}^{rw \geq 0}$).

Now we denote Y_0 by Y and X_0 by Z . Y, Z will be our candidates for $X^{w \geq 0}$ and $X^{w \leq 0}$.

It remains to prove that $Z \in \underline{C}^{rw \leq 0}$. We should check that $\underline{C}(C, Z[k]) = \{0\}$ for all $k > 0$, $C \in \text{Obj} H$. Since $\underline{C}(-, Z)$ transforms arbitrary coproducts into products, it suffices to consider $C \in \text{Obj} H'$.

First we prove that $C \perp X_{k-1}[k]$ for all $k \leq j$.

We use the distinguished triangle

$$V_k : P_k \rightarrow X_k[k] \rightarrow X_{k-1}[k] \rightarrow P_k[1]. \quad (22)$$

Using (21), we obtain $\underline{C}(C, P_k[k]) = \bigoplus_{Y \in \text{Obj} H', f_i(Y, X_k[k])} \underline{C}(C, Y)$. By the definition of $f_i(C, Y)$ we obtain that this group surjects onto $\underline{C}(C, X_k[k])$. Moreover, $\underline{C}(C, P_k[1]) = \bigoplus_{Y \in \text{Obj} H', f_i(Y, X_k[k])} \underline{C}(C, Y[1]) = \{0\}$. We obtain $\underline{C}(C, X_{k-1}[k]) = \{0\}$.

Now we use distinguished triangles V_l for all $l < k$. Again (21) yields $\underline{C}(C, P_l[1]) = \underline{C}(C, P_l[2]) = \{0\}$. Hence $\underline{C}(C, X_{l-1}[k]) = \underline{C}(C, X_l[k]) = \{0\}$ for all $l < k$.

Hence $C \perp Z[k]$ for all $j \leq k > 0$.

Lastly, the distinguished triangles V_k easily yield by induction that $C \perp X_l[k]$ for all $l \leq j$ and $k > j$.

The proof in case (ii) is almost the same; one should only always replace some choice of generators $f_i(Y, Z) \in \underline{C}(Y, Z)$ by all elements of $\underline{C}(Y, Z)$.

(\underline{C}, w) is obviously bounded above by (20).

Now we check that (\underline{C}, w) is non-degenerate. The condition (20) implies that $\cap \underline{C}^{rw \geq i} = \{0\}$. Next, for any $X \in \text{Obj} \underline{C} \setminus \{0\}$ there exists an $f \in \underline{C}(Y[i], X)$ for some $Y \in H$ and $i \in \mathbb{Z}$ such that $f \neq 0$. Hence such an X does not belong to $\underline{C}^{rw \leq -1-i}$ (see the definition of $\underline{C}^{rw \leq i}$ in the proof of part III).

It remains to consider $\underline{H}w$. It contains H by the definition of w . The method used in the proof of assertion II2 also easily yields that any object of $\underline{H}w$ is a retract of an object of H .

IV Everything is obvious except that a cone of a morphism of objects of \underline{C}^- belongs to $\text{Obj} \underline{C}^-$. This fact is easy also since the functors $\underline{C}(Y, -)$ are homological.

□

Corollary 4.3.3. *It is well known that there are no morphisms of positive degrees between (copies) of the sphere spectrum S^0 in the stable homotopy category SH ; cf. §4.6 below. Hence part II of Theorem 4.3.2 immediately implies: the category of finite spectra SH_{fin} (i.e. the full subcategory of SH generated by S^0) has a bounded weight structure w . Its heart can be described as a category H of finite sums of (copies of) S^0 (since any retract of $(S^0)^n$, $n > 0$, is trivial, no new objects appear in the small envelope of H). Since $SH(S^0, S^0) = \mathbb{Z}$, $\underline{H}w$ is equivalent to $Ab_{fin, fr}$ (the category of finitely generated free abelian groups).*

This weight structure obtained is adjacent to the Postnikov t -structure for SH ; see Definition 4.4.1 and §4.6 below.

Remark 4.3.4. 1. Recall that in the proof of Theorem 4.3.2 it was specified explicitly how to recover w from $\underline{C}^{w=0}$.

2. The conditions of parts III and IV of the theorem could seem to be rather exotic. Yet they can be easily verified for a certain subcategory of *quasi-finite* objects in SH , see §4.6.

3. Obviously, if for any $X \in \text{Obj} \underline{C}$ and any $Y \in \text{Obj} H'$ the group $\underline{C}(Y, X)$ is generated by $< c$ elements as a group, then it is also generated by $< c$ elements as a $\underline{C}(Y, Y)$ -module. In particular, this is the condition which we will actually check for the category SH_{fin} .

4. $\underline{C}^{w \leq 0}$ described in the proof of part III of Theorem 4.3.2 is often a $\underline{C}^{t \leq 0}$ -part of a certain t -structure; then this t -structure is *left adjacent* to w (see Definition 4.4.1 below). Yet in order for the t -decompositions to exist when we take the only possible candidate for $\underline{C}^{t \geq 0}$ (cf. Proposition 4.4.7 below) the homotopy colimit of all Y_i (defined as in the proof of part III) should exist for all $X \in \text{Obj} \underline{C}$ (cf. the proof of Proposition 4.5.2). Note that this is not true for the category SH_{fin} (see Corollary 4.3.3). For example, one can note that Eilenberg-MacLane spectra do not belong to SH_{fin} . Another example: one can define the Chow weight structure on DM_{gm}^{eff} whereas the corresponding t -structure is only defined on DM_-^{eff} ; see §6.5 and §7.1.

This shows that weight structures exist 'more often' than t -structures, whereas Theorem 4.4.2 below shows that they contain (more or less) the same information as the corresponding t -structures. This evidence supports the author's belief that weight structures are more relevant for 'general' triangulated categories than t -structures. See also (part 2 of) Remark 2.5.2 for more facts supporting this opinion.

Besides, it can be easily seen that the natural 'almost dual' to the statement of part II (i.e. we take a positive generating subcategory H and ask whether a t -structure with $H \subset \underline{H}t$ exists) is false.

4.4 Adjacent weight and t -structures; weight-exact functors

Definition 4.4.1. We say that a weight structure w is left (resp. right) *adjacent* to a t -structure t if $\underline{C}^{w \leq 0} = \underline{C}^{t \leq 0}$ (resp. $\underline{C}^{w \geq 0} = \underline{C}^{t \geq 0}$).

In this situation we will also say that t is adjacent to w .

A simple example is: \underline{C} is (an appropriate version of) $D(A)$ for an abelian A . Then (under the conditions that ensure that $D^?(A)$ is isomorphic to $K^?(Proj A)$) the stupid weight structure coming from $K^?(Proj A)$ is left adjacent to the canonical t -structure for $D^?(A)$. Dually, if $D^?(A) \cong K^?(Inj A)$, then the canonical t -structure for it is right adjacent to the stupid weight structure coming from $K^?(Inj A)$. We will construct more (interesting) examples of adjacent structures below.

The following theorem describes several properties of adjacent structures. Parts 1,2 of the theorem were already proved (above); parts 7,8 of it show that for $Y \in Obj \underline{C}$ the virtual t -truncations of the functor $\underline{C}(-, Y)$ (defined in Remark 2.5.2) are represented by actual t -truncations of Y with respect to t . We also prove that \underline{Ht} and \underline{Hw} are connected by a natural generalization of the relation between the categories A and $Proj A$ (for an abelian A). Still note that in the latter case we also have $\underline{Hw} \subset \underline{Ht}$; this is a rather non-typical situation.

Theorem 4.4.2 (Duality theorem). *Let w be left adjacent to t . Then the following statements are valid.*

1. $\underline{C}^{w \geq 0} = {}^\perp \underline{C}^{w \leq -1}$; $\underline{C}^{w \leq 0} = (\underline{C}^{w \geq 1})^\perp$.
2. $\underline{C}^{t \geq 0} = (\underline{C}^{t \leq -1})^\perp$; $\underline{C}^{t \leq 0} = {}^\perp \underline{C}^{t \geq 1}$.
3. The functor $\underline{C}(-, \underline{Ht}) : \underline{Hw} \rightarrow \underline{Ht}^*$ (see the Notation) that sends $X \in \underline{C}^{w=0}$ to $Y \rightarrow \underline{C}(X, Y)$, ($Y \in \underline{C}^{t=0}$), is a full embedding of \underline{Hw} into the full subcategory $Ex(\underline{Ht}, Ab) \subset \underline{Ht}^*$ which consists of exact functors.
4. The functor $\underline{C}(-, \underline{Ht}) : \underline{Ht} \rightarrow \underline{Hw}_*$ that sends $X \in \underline{C}^{t=0}$ to $Y \mapsto \underline{C}(Y, X)$, ($Y \in \underline{C}^{w=0}$), is a full exact embedding of \underline{Ht} into the abelian category \underline{Hw}_* .
5. Let t be non-degenerate. Then $\underline{C}^{t=0}$ equals $S = \{X \in Obj \underline{C} : \underline{C}^{w=0} \perp X[i] \ \forall i \neq 0\}$.
6. Let $i \in \mathbb{Z}$, let $Y \in Obj \underline{C}$ be fixed. Then for the functor $F(X) = \underline{C}(X, Y)$ we have $W^i(F)(X) = \text{Im}(\underline{C}(X, \tau_{\leq i} Y) \rightarrow \underline{C}(X, Y))$ for any $X \in Obj \underline{C}$.
7. For any i, j we have a functorial isomorphism

$$\underline{C}(X, Y^{t \leq i}[j]) \cong \text{Im}(\underline{C}(X^{w \leq -j}, Y[i]) \rightarrow \underline{C}(X^{w \leq 1-j}, Y[i+1])).$$

8. For any i, j we have a functorial isomorphism

$$\underline{C}(X, Y^{t \geq i}[j]) \cong \text{Im}(\underline{C}(X^{w \geq -j}, Y[i]) \rightarrow \underline{C}(X^{w \geq -1-j}, Y[i-1])).$$

9. For any $X, Y \in \text{Obj } \underline{C}$ let $\cdots \rightarrow Y^{-1} \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots$ denote an arbitrary choice of the weight complex for Y (in its homotopy equivalence class). Then we have

$$\underline{C}(Y, X^{t=0}) = (\text{Ker}(\underline{C}(Y^0, X) \rightarrow \underline{C}(Y^{-1}, X)) / \text{Im}(\underline{C}(Y^1, X) \rightarrow \underline{C}(Y^0, X))). \quad (23)$$

10. We have $\underline{C}^{b, w \geq 0} = \text{Obj } \underline{C}^b \cap (\perp \cup_{i < 0} \underline{C}^{t=i})$ and $\underline{C}^{b, w \leq 0} = \text{Obj } \underline{C}^b \cap (\perp \cup_{i > 0} \underline{C}^{t=i})$.

Proof. 1. This is just parts 1 and 2 of Proposition 1.3.3.

2. A well-known property of t -structures (certainly, it doesn't depend on w).

3. First we note that for any $X \in \underline{C}^{w=0}$ orthogonality for w implies $X \perp \underline{C}^{w \leq -1} (= \underline{C}^{t \leq -1})$, whereas orthogonality for t gives $X \perp \underline{C}^{t \geq 1}$. In particular,

$$X \perp (\cup_{i \neq 0} \underline{C}^{t=i}). \quad (24)$$

Now, short exact sequences in \underline{Ht} give distinguished triangles in \underline{C} . Hence for any homological functor $F : \underline{C} \rightarrow \text{Ab}$ and for $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ being a short exact sequence in \underline{Ht} we have a long exact sequence $\cdots \rightarrow F(C[-1]) \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow F(A[1]) \rightarrow \cdots$. If $F = \underline{C}(X, -)$ then $F(C[-1]) = F(A[1]) = \{0\}$ (as was just noted). Hence objects of \underline{Hw} induce exact functors on \underline{Ht} .

To prove that the restriction $\underline{Hw} \rightarrow \underline{Ht}^*$ is a fully faithful functor it suffices to prove that the restriction of the functor $\underline{C}(X, -)$ to \underline{Ht} for $X \in \underline{Hw}$ determines X in a functorial way. Using Yoneda's lemma, we see that it suffices to recover $\underline{C}(-, X)$ from its restriction.

We prove that

$$\underline{C}(X, Y) \cong \underline{C}(X, H_t^0(Y)) \quad \forall X \in \underline{C}^{w=0}, Y \in \text{Obj } \underline{C}. \quad (25)$$

We apply t -decompositions (i.e. (18)) twice.

First we obtain a distinguished triangle $Y^{t \geq 1}[-1] \rightarrow Y^{t \leq 0} \rightarrow Y \rightarrow Y^{t \geq 1}$. Since $\underline{C}(X, Y^{t \geq 1}[-1]) = \underline{C}(X, Y^{t \geq 1})$, we obtain $\underline{C}(X, Y) = \underline{C}(X, Y^{t \leq 0})$.

Next, we have a distinguished triangle $Y^{t \leq -1} \rightarrow Y^{t \leq 0} \rightarrow H_t^0(Y) \rightarrow Y^{t \leq -1}[1]$. Since $\underline{C}(X, Y^{t \leq -1}) = \underline{C}(X, Y^{t \leq -1}[1]) = \{0\}$, we obtain (25).

4. Again, it suffices to prove that the restriction of the functor $\underline{C}(-, X)$ to \underline{Hw} for $X \in \underline{C}^{t=0}$ determines X functorially.

We note that for any $X \in \underline{C}^{t=0}(\subset \underline{C}^{w \leq 0})$ the orthogonality axiom for t implies $(\underline{C}^{w \leq -1}) = \underline{C}^{t \leq -1} \perp X$, whereas orthogonality for w gives $\underline{C}^{w \geq 1} \perp X$.

Now we prove that

$$\underline{C}(Y, X) \cong (\text{Ker}(\underline{C}(Y^0, X) \rightarrow \underline{C}(Y^{-1}, X)) / \text{Im}(\underline{C}(Y^1, X) \rightarrow \underline{C}(Y^0, X))). \quad (26)$$

Indeed, consider the (infinite) weight decomposition of Y that gives our choice of the weight complex and apply Theorem 2.4.2 to the functor $\underline{C}(-, X)$. The spectral sequence obtained converges since it satisfies condition II(ii) of Theorem 2.4.2 (it has only one non-zero column!). It remains to note that this (possibly) non-zero column of $(E_1^{pq}(T(\underline{C}(-, X)(Y)))) = (\underline{C}(Y^{-p}, X[q]))$ is exactly $(E_1^{p0}(T)) = \dots \rightarrow \underline{C}(Y^1, X) \rightarrow \underline{C}(Y^0, X) \rightarrow \underline{C}(Y^{-1}, X) \rightarrow \dots$.

We obtain (26).

5. By assertion 4, an object of $\underline{H}t$ is non-zero if and only if it represents a non-zero functor on $\underline{H}w$. Hence applying (25) we obtain that S is exactly the class of objects that satisfy $H^{it}(X) = 0$ for all $i \neq 0$. It remains to note that for a non-degenerate t this class is exactly $\underline{C}^{t=0}$.

6. We can assume that $i = 0$. We should check that $g \in \underline{C}(X, Y)$ lifts to some $h \in \underline{C}(w_{\leq 0}X, Y)$ whenever it lifts to some $l \in \underline{C}(X, \tau_{\leq 0}Y)$. Now, the equality

$$\underline{C}(w_{\leq 0}X, \tau_{\geq 1}Y) = \underline{C}(w_{\geq 1}X, \tau_{\leq 0}Y) = \{0\}$$

yields that any morphism of any of the two morphism groups in question can be lifted to some $m \in \underline{C}(w_{\leq 0}X, \tau_{\leq 0}Y)$. Hence if one of (h, l) exists, then the other one can be constructed from the corresponding m using the commutativity of:

$$\begin{array}{ccc} \underline{C}(w_{\leq 0}X, \tau_{\leq 0}Y) & \longrightarrow & \underline{C}(w_{\leq 0}X, Y) \\ \downarrow & & \downarrow \\ \underline{C}(X, \tau_{\leq 0}Y) & \longrightarrow & \underline{C}(X, Y) \end{array}$$

7. Shifting X, Y we can easily reduce the statement to the case $i = j = 0$.

The t -decomposition of Y yields exact sequences $\{0\} = \underline{C}(X^{w \leq 0}, Y^{t \geq 1}[-2]) \rightarrow \underline{C}(X^{w \leq 0}, Y^{t \leq 0}) \rightarrow \underline{C}(X^{w \leq 0}, Y) \rightarrow \underline{C}(X^{w \leq 0}, Y^{t \geq 1}[-1]) = \{0\}$ and

$$\{0\} = \underline{C}(X^{w \leq 1}, Y^{t \geq 1}[-1]) \rightarrow \underline{C}(X^{w \leq 1}, Y^{t \leq 0}[1]) \rightarrow \underline{C}(X^{w \leq 1}, Y[1]) \rightarrow \underline{C}(X^{w \leq 1}, Y^{t \geq 1}) \rightarrow \dots$$

Next, weight decompositions of X and $X[1]$ similarly yield that the obvious homomorphism $\underline{C}(w_{\leq 0}X, Y^{t \leq 0}) \rightarrow \underline{C}(w_{\leq 1}X, Y^{t \leq 0})$ is surjective whereas $\underline{C}(w_{\leq 1}X, Y^{t \leq 0}) \cong \underline{C}(X, Y^{t \leq 0})$.

We obtain a commutative diagram

$$\begin{array}{ccc} \underline{C}(w_{\leq 0}X, Y^{t \leq 0}) & \xrightarrow{f} & \underline{C}(w_{\leq 0}X, Y) \\ \downarrow g & & \downarrow h \\ \underline{C}(w_{\leq 1}X, Y^{t \leq 0}) & \xrightarrow{p} & \underline{C}(w_{\leq 1}X, Y) \end{array}$$

with f being bijective, g being surjective, and p being injective. Hence $\underline{C}(X, Y^{t \leq 0}) \cong \underline{C}(w_{\leq 1}X, Y^{t \leq 0}) \cong \text{Im } g \cong \text{Im } h$.

Note that the isomorphism constructed is obviously natural in Y whereas it is natural in X by part 2 of Lemma 1.5.1.

8. This assertion is exactly the dual of the previous one (see Remark 1.1.2).

9. Immediate from (26) and (25).

10. Let $X \in \text{Obj } \underline{C}^b$.

If we also have $X \in \underline{C}^{w \geq 0}$ (resp. $X \in \underline{C}^{w \leq 0}$) the orthogonality statements desired are valid by assertion 1.

Now we prove the converse implication. Let $X \perp \cup_{i < 0} \underline{C}^{t=i}$. We should check that $X \perp Z$ for all $Z \in \underline{C}^{t \leq -1}$. We have $X \perp (H_t^i(Z)[-i])$ for all i . Besides, since X is bounded, we have $X \perp (\tau_{\leq j}Z)$ for some j (that is small enough). Hence considering the t -decompositions of $Z^{t \leq k}[-1]$ for all $k > j$ one can easily obtain the orthogonality statement required.

The case $X \perp \cup_{i > 0} \underline{C}^{t=i}$ is considered similarly.

□

Remark 4.4.3. 1. One can often describe the images of embeddings in parts 3 and 4.

2. Dually to assertion 6: for w right adjacent to t and $F(X) = \underline{C}(Y, X)$ we obtain $W_i(F)(X) = \text{Im}(\underline{C}(\tau_{\geq i}Y, X) \rightarrow \underline{C}(Y, X))$.

3. Assertion 6 and its dual show that weight truncations are 'almost adjoint' to the corresponding t -truncations. These statements are counterparts to the fact that (for an arbitrary t -structure) any morphism $X \rightarrow Y$ for $Y \in \underline{C}^{t \geq 0}$ can be uniquely factorized through $X^{t \geq 0}$ (and to its dual).

4. Assertions 7 and 9 (almost) imply that the derived exact couple for the spectral sequence $\underline{C}(X^{-p}[-q], Y) \implies \underline{C}(X[-p-q], Y)$ (as in Theorem 2.4.2) can also be described in terms of $\underline{C}(X[i], Y^{t \leq j})$ and $\underline{C}(X[i], Y^{t=j})$; see §2.6 of [Bon10] for the complete proof of this fact. This is no surprise by parts 3,4 of Remark 2.5.2. It follows that the spectral sequence S converging to $\underline{C}(X, Y)$ corresponding to the t -truncations of Y can be 'embedded' into our T (i.e. for all $i > 0$ any $E_i^{p,q}(S) \cong E_{i+1}^{p',q'}(T)$ for $p' = q + 2p, q' = -p$; these isomorphisms respect the structure of spectral sequences).

In algebraic topology, this statement corresponds to the fact (and implies it) that the Atiyah-Hirzebruch spectral sequence for the cohomology of a space X with coefficients in a spectrum S can be obtained either by considering the cellular filtration of X or the Postnikov t -truncations of S .

Note that in our method we describe all terms of exact couples (in contrast to [Par96], for example). The advantage of this is that the D -terms of those could be very interesting; see Proposition 7.4.2 and Corollary 7.5.2 below.

5. In fact, one can extend the notion of adjacent structures to the case when there are two distinct triangulated categories \underline{C} with a weight structure w and \underline{D} with a t -structure t . They should be equipped with a *duality* $\Phi : \underline{C}^{op} \times \underline{D} \rightarrow A$, A is an abelian category (that generalizes $\underline{C}(-, -) : \underline{C}^{op} \times \underline{C} \rightarrow Ab$) along with a (bi)natural transformation $\Phi(X, Y) \cong \Phi(X[1], Y[1])$; Φ is homological with respect to both arguments (see Definition 2.5.1 of [Bon10]). $\underline{D}^{t \geq 1}$ should annihilate $\underline{C}^{w \leq 0}$ with respect to Φ , whereas $\underline{D}^{t \leq -1}$ should annihilate $\underline{C}^{w \geq 0}$; cf. (part 1 of) Remark 2.5.2. In Theorem 2.6.1 of [Bon10] the author proves natural analogues of parts 6–9 of Theorem 4.4.2. This yields another proof of the comparison of spectral sequences statement of §6.4 (in the general case; see Remark 6.4.1); cf. also (parts 3 and 4 of) Remark 2.5.2.

See also §8.3 below for further ideas in this direction.

6. Even more generally, for parts 7,8 of the theorem it suffices for X, Y (lying either in the same category or in different ones) to have Postnikov towers whose terms satisfy the same orthogonality conditions as those provided by the definition of the weight structure. Indeed, the same proofs work!

Furthermore, it is sufficient for the orthogonality conditions to be satisfied ‘in the limit’ for a directed system of Postnikov towers (for X). Again, it is no problem to generalize the proof to this case. Still, in order to make the statement easier to understand, the author chose to formulate it in §7.4 below only for a partial (yet very important!) case corresponding to coniveau spectral sequences.

Recall now: if an exact functor $\underline{C} \rightarrow \underline{C}'$ is t -exact with respect to some t -structures on these categories, its (left or right) adjoint is usually not t -exact (it is only left or right t -exact, respectively). This situation can be described much more precisely if there exist adjacent weight structures for these t -structures.

Till the end of this subsection, \underline{C} and \underline{C}' will be triangulated categories, that are possibly endowed with t -structures t and t' , and with weight structures w and w' , respectively. $F : \underline{C} \rightarrow \underline{C}'$ and $G : \underline{C}' \rightarrow \underline{C}$ will be exact functors.

Definition 4.4.4. 1. F is called t -exact (with respect to t, t') if it maps $\underline{C}^{t \leq 0}$ to $\underline{C}^{t' \leq 0}$, and maps $\underline{C}^{t \geq 0}$ to $\underline{C}^{t' \geq 0}$.
 2. F is called weight-exact (with respect to w, w') if it maps $\underline{C}^{w \leq 0}$ to $\underline{C}^{w' \leq 0}$, and maps $\underline{C}^{w \geq 0}$ to $\underline{C}^{w' \geq 0}$.

Certainly, t - and weight-exact functors also map hearts of the corresponding structures to hearts.

Now we describe the main properties of the notions introduced.

Proposition 4.4.5. 1. Any composition of t -exact functors is t -exact.
 2. Any composition of weight-exact functors is weight-exact.
 3. Let (w, t) and (w', t') be left adjacent; suppose that G is left adjoint to F . Then G is weight-exact if and only if F is t -exact.
 4. Let (w, t) and (w', t') be right adjacent; suppose that G is right adjoint to F . Then G is weight-exact if and only if F is t -exact.

Proof. 1. Obvious from the definition of t -exact functors, and also well-known.

2. Obvious from the definition of weight-exact functors.

3. By part 1 of Theorem 4.4.2, $G(\underline{C}^{w' \geq 0}) \subset \underline{C}^{w \geq 0}$ whenever $G(\underline{C}^{w' \geq 0}) \perp \underline{C}^{w \leq -1}$. The adjunction yields (after shifting by $[-1]$): this is equivalent to $\underline{C}^{w' \geq 1} \perp F(\underline{C}^{w \leq 0})$. Applying part 1 of Theorem 4.4.2 again, we obtain that this is equivalent to $F(\underline{C}^{w \leq 0}) \subset \underline{C}^{w' \leq 0}$. Now recall that $\underline{C}^{w \leq 0} = \underline{C}^{t \leq 0}$ and $\underline{C}^{w' \leq 0} = \underline{C}^{t' \leq 0}$. Hence we proved that $G(\underline{C}^{w' \geq 0}) \subset \underline{C}^{w \geq 0}$ if and only if $F(\underline{C}^{t \leq 0}) \subset \underline{C}^{t' \leq 0}$.

It remains to check that $G(\underline{C}^{w' \leq 0}) \subset \underline{C}^{w \leq 0}$ whenever $F(\underline{C}^{t \geq 0}) \subset \underline{C}^{t' \geq 0}$. Essentially, this is a well-known property of t -structures (that we already mentioned above). The first inclusion formula can be rewritten as: $G(\underline{C}^{t' \leq 0}) \subset \underline{C}^{t \leq 0}$. Next, part 2 of Theorem 4.4.2 yields that this happens whenever $G(\underline{C}^{t' \leq 0}) \perp \underline{C}^{t \geq 1}$. Applying the adjunction and shift by $[1]$ we conclude the proof.

4. This is exactly the dual of the previous assertion (see Remark 1.1.2). \square

Remark 4.4.6. Certainly, one can also introduce the notions of right weight-exact and left weight-exact functors, and prove the natural versions of the assertions above for them. In fact, the arguments needed for this are essentially contained in the proof of Proposition 4.4.5.

Theorem 4.4.2 also yields a simple description of adjacent structures (of any type) when they exist.

Proposition 4.4.7. 1. Let w be a weight structure for \underline{C} . Then there exists a t -structure which is left (resp. right) adjacent to w if and only if for $\underline{C}^{t \leq 0} = \underline{C}^{w \leq 0}$ and $\underline{C}^{t \geq 0} = (\underline{C}^{w \leq -1})^\perp$ (resp. $\underline{C}^{t \geq 0} = \underline{C}^{w \geq 0}$ and $\underline{C}^{t \leq 0} = {}^\perp \underline{C}^{w \geq 1}$), and any $X \in \text{Obj} \underline{C}$ there exists a t -decomposition (18) of X . In this case our choice of $(\underline{C}^{t \leq 0}, \underline{C}^{t \geq 0})$ is the only one possible.

2. Let t be a t -structure for \underline{C} . Then there exists a weight structure which is left (resp. right) adjacent to w if and only if for $\underline{C}^{w \leq 0} = \underline{C}^{t \leq 0}$ and $\underline{C}^{w \geq 0} = {}^\perp \underline{C}^{t \leq -1}$ (resp. $\underline{C}^{w \geq 0} = \underline{C}^{t \geq 0}$ and $\underline{C}^{w \leq 0} = (\underline{C}^{t \geq 1})^\perp$), and any $X \in \text{Obj} \underline{C}$ there exists a weight decomposition (1) of X . In this case our choice of $(\underline{C}^{w \leq 0}, \underline{C}^{w \geq 0})$ is the only one possible.

Proof. First we note that by parts 1,2 of Theorem 4.4.2 our choices of the structures are the only ones possible. Hence it suffices to check when these choices indeed give the corresponding structures.

1. We only consider the left adjacent structure case; the 'right case' is similar (and, in fact, dual; see Remark 1.1.2).

The class $\underline{C}^{t \geq 0}$ is automatically strict; since $\underline{C}^{w \leq 0}[1] \subset \underline{C}^{w \leq 0}$, we have $\underline{C}^{t \geq 0} \subset \underline{C}^{t \geq 0}[1]$.

Hence we obtain a t -structure if and only if there always exist t -decompositions.

2. As in part 1, we consider only the 'left' case (for the same reason).

It is well known that $\underline{C}^{t \leq 0}$ is Karoubi-closed; hence both $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ are Karoubi-closed also. Again $\underline{C}^{t \leq 0}[1] \subset \underline{C}^{t \leq 0}$ implies $\underline{C}^{t \geq 0} \subset \underline{C}^{t \geq 0}[1]$.

Hence we obtain a weight structure if and only if there always exist weight decompositions. \square

4.5 Existence of adjacent structures

Now we study certain sufficient conditions for adjacent weight and t -structures to exist. We will prove a statement that is relevant for Voevodsky's DM_-^{eff} and for SH .

First we describe a certain version of the compactly generated category notion; DM_-^{eff} and SH will satisfy our conditions.

Definition 4.5.1. We will say that a set of objects $C_i \in \text{Obj} \underline{C}$, $i \in I$ (I is a set) *negatively well-generates* \underline{C} if

- (i) C_i are compact; they weakly generate \underline{C} (cf. the Notation).
- (ii) For all $j > 0$ we have $\{C_i\} \perp \{C_i[j]\}$ (i.e. the set $\{C_i\}$ is negative).
- (iii) \underline{C} contains the category H whose objects are arbitrary (small) co-products of C_i ; \underline{C} also contains all homotopy colimits of $X_i \in \text{Obj} \underline{C}$ (see Definition 4.2.1) such that $X_{-1} = 0$ and $\text{Cone}(X_i \rightarrow X_{i+1}) \in \text{Obj} H[i]$.

Theorem 4.5.2. *I1. Suppose that $C_i \in \text{Obj} \underline{C}$, $i \in I$, negatively well-generate \underline{C} . For H described in (iii) of Definition 4.5.1 we consider a full subcategory $\underline{C}^- \subset \underline{C}$ whose objects are*

$$X \in \text{Obj} \underline{C}^- : \forall Y \in \text{Obj} H \text{ there exists } j \in \mathbb{Z} \text{ such that } Y \perp X[i] \forall i > j. \quad (27)$$

Then there exist a weight structure w on \underline{C}^- and a t -structure t on \underline{C} such that $H \subset \underline{Hw}$, t restricts to a t -structure on \underline{C}^- , and $\underline{C}^{t \leq 0} = \underline{C}^{-, w \leq 0}$.

(Note that w and t restricted to \underline{C}^- are adjacent by definition.)

2. If \underline{C} also admits arbitrary countable coproducts, then w can be extended to the whole \underline{C} .

II Let C_i, \underline{C}, w, t be either as in assertion I2 or as in I1 with the additional condition $\underline{C} = \underline{C}^-$ (i.e. w is defined on \underline{C}) fulfilled. Then the following statements are valid.

1. \underline{Hw} is the idempotent completion of the category H (whose objects are coproducts of C_i) in \underline{C} .

2. Restrict the functors from \underline{Ht} (considered as a subset of \underline{Hw}_ by part 4 of Theorem 4.4.2) to the full additive subcategory $C \subset \underline{Hw}$ consisting of finite coproducts of C_i . Then this restriction functor gives an equivalence of \underline{Ht} with C_* .*

3. For any object of $Y \in \underline{C}^{t=0}$ and any $X \in \text{Obj} \underline{C}$ we have $\underline{C}(X, Y) = (\text{Ker}(\underline{C}(X^0, Y) \rightarrow \underline{C}(X^{-1}, Y)) / \text{Im}(\underline{C}(X^1, Y) \rightarrow \underline{C}(X^0, Y)))$ where $\cdots \rightarrow X^{-1} \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots$ is an arbitrary choice of a weight complex for X .

Proof. I1. The existence of w on \underline{C}^- is immediate from part III (version (ii)) of Theorem 4.3.2.

Now define $\underline{C}^{t \geq 0} = (\underline{C}^{w \leq -1})^\perp$. In order to prove that t is a t -structure it suffices (cf. Proposition 4.4.7) to check that for any $X \in \text{Obj} \underline{C}$ there exists a t -decomposition (18).

We will construct $X^{t \leq 0}$ and $X^{t \geq 1}$ explicitly. Our construction uses almost the same argument as the one in the proof of part III version (ii) of Theorem 4.3.2. It could also be thought about as of a triangulated version of the construction of Eilenberg-MacLane spaces (this construction really allows to construct Eilenberg-MacLane spectra from S^0 in SH , see §4.6 below!).

We take $P_0 = \coprod_{i \in I, s \in \underline{C}(C_i, X)} C_i$. Then we have a morphism $f_0 : P_0 \rightarrow X$ whose component that corresponds to (C_i, s) is given by s . Let X_0 denote a cone of f_0 . Repeating the construction for $X_0[-1]$ instead of X we get an object P_1 being a coproduct of certain C_i , $f_1 : P_1 \rightarrow X_0[-1]$; we denote a cone of f_1 by $X_1[-1]$. Proceeding (with $X_i[-1-i]$), we get an infinite sequence of (P_i, f_i, X_i) . We denote the map $X \rightarrow X_0$ given by the construction by g_0 ,

$g_i : X_{i-1} \rightarrow X_i$, $h_i = g_i \circ \cdots \circ g_1 \circ g_0 : X \rightarrow X_i$. We denote a cone of h_i by $Y_i[1]$; the map $Y_i \rightarrow X[1]$ given by the corresponding distinguished triangle by r_i . We have $P_i \in \underline{C}^{w=0}$ by the definition.

We have $Y_0 = P_0$. Then part 2 of Remark 1.5.9 yields that $Y_i \in \underline{C}^{w \leq 0}$ for all i .

Now we consider the homotopy colimit of Y_i ; cf. Definition 4.2.1. By part 1 of Lemma 4.2.3, the sequence r_i can be lifted to some morphism $f : Y \rightarrow X$. We denote its cone as Z .

Since (\underline{C}, w) admit negative coproducts, by part 3 of Remark 4.2.2 we have $Y \in \underline{C}^{w \leq 0} (= \underline{C}^{t \leq 0})$. Y, Z will be our candidates for $X^{t \leq 0}$ and $X^{t \geq 1}$.

We verify that $Z \in \underline{C}^{t \geq 1}$. First we check that $C_i[j] \perp Z$ for all $i \in I$, $j \geq 0$. This is equivalent to the fact that the map $f_* : \underline{C}(C_i[j], Y) \rightarrow \underline{C}(C_i[j], X)$ is an isomorphism for all $i \in I$, $j \geq 0$ and is injective for $j = -1$ (cf. part 1 of Remark 1.3.4).

By part 3 of Lemma 4.2.3, for any compact C we have $\underline{C}(C, Y) = \varinjlim \underline{C}(C, Y_i)$. Moreover, we have $C_i[-1] \perp Y$ since $Y \in \underline{C}^{w \leq 0}$ and $C_i[-1] \in \underline{C}^{w=1}$. Hence it suffices to verify that $C_i[j] \perp X_l$ for $l > j \geq 0$ (this gives $\underline{C}(C_i[j], Y_l) \cong \underline{C}(C_i[j], X)$).

We apply the distinguished triangle $P_j[j] \rightarrow X_{j-1} \rightarrow X_j \rightarrow P_j[j+1]$. Since $C_i[j]$ is compact, we easily obtain

$$\underline{C}(C_i[j], P_j[j]) = \bigoplus_{m \in I, s \in \underline{C}(C_m[j], X_{j-1})} \underline{C}(C_i, C_m).$$

Hence this group has an element for each morphism $C_m[j] \rightarrow X_{j-1}$; it follows that the map $\underline{C}(C_i[j], P_j[j]) \rightarrow \underline{C}(C_i[j], X_{j-1})$ is surjective. Next (since $C_i[j]$ is compact and $C_i[j] \in \underline{C}^{w=j}$), $\underline{C}(C_i[j], P_j[j+1]) = \underline{C}(C_i, P_j[1])$ equals the direct sum of the corresponding $\underline{C}(C_i, C_m[1])$; hence it is zero by the orthogonality property for w (cf. Definition 1.1.1). We obtain $C_i[j] \perp X_j$.

Now we consider distinguished triangles $P_l[l] \rightarrow X_{l-1} \rightarrow X_l \rightarrow P_l[l+1]$ for $l > j$. Again compactness of $C_i[j]$ yields $\underline{C}(C_i[j], P_l[l+1]) = \underline{C}(C_i[j], P_l[l]) = \{0\}$. Hence $\underline{C}(C_i[j], X_l) = \underline{C}(C_i[j], X_{l-1}) = \{0\}$ for all $l > j$.

It remains to check that for any $T \in \text{Obj } \underline{C}$ the condition $\{C_i[j]\} \perp T$ for all $j \geq 0$ implies that $\underline{C}^{w \leq 0} \perp T$. This follows immediately from part III (version (ii)) of Theorem 4.3.2.

Moreover, loc.cit. also implies that on \underline{C}^- (defined as in part IV of loc.cit.) there exists a weight structure such that $H \subset \underline{H}w$. Note that \underline{C}^- also satisfies the conditions of the theorem. The description of $\underline{C}^{w \leq 0}$ in the proof of Theorem 4.3.2 shows that w is left adjacent to t on \underline{C}^- .

2. We should check that w can be extended to the whole \underline{C} . We define $X^{w \geq 1}$ using the orthogonality axiom (of weight structures).

For any $X \in \text{Obj} \underline{C}$ we denote $\tau_{\leq i} X$ by X_i for $i > 1$ and take Y being the homotopy colimit of Y_i for $Y_i = X_i^{w \geq 1}$ (see Definition 4.2.1). Here the morphisms $Y_i \rightarrow Y_{i+1}$ are obtained by applying part 1 of Lemma 1.5.1 to the natural morphisms $X_i \rightarrow X_{i+1}$. By Lemma 4.2.4 we can assume that $Y \in \underline{C}^{w \geq 0}$.

Y will be our candidate for $X^{w \geq 1}$ (cf. the proof of part II). By part 1 of Lemma 4.2.3 the system of composition maps $Y_i[-1] \rightarrow X_i \rightarrow X$ can be lifted to some $f \in \underline{C}(Y[-1], X)$.

Now we show that f extends to a weight decomposition of X using part 1 of Remark 1.3.4. We should check that $C_k[j] \perp \text{Cone}(f)$ for all $k \in I$ and $j < 0$ (see the description of $\underline{C}^{w \leq 0} = \underline{C}^{-, w \leq 0}$ in the proof of part III of Theorem 4.3.2). Since all C_k are compact, as in the proof of part II we obtain that $\underline{C}(C_k[j], Y) = \varinjlim \underline{C}(C_k[j], Y_i)$. Moreover, $\underline{C}(C_k[j], X_i) = \underline{C}(C_k[j], X)$ for $i > -j$. Hence it suffices to note that the direct limit of isomorphisms is an isomorphism, whereas the direct limit of surjections is surjective if the targets stabilize (obvious!).

Now it remains to apply part 4 of Lemma 1.3.7.

II1. Obviously, \underline{Hw} contains $\text{Obj} H$. Since \underline{Hw} is Karoubi-closed in \underline{C} , it also contains all retracts of objects of H .

We check that any idempotent $h \in \underline{C}(X, X)$ yields an object of \underline{C} if $X \in \text{Obj} H$. We apply Neeman's argument (see Proposition 1.6.8 of [Nee01]). One can easily check that the (formal) image of h can be presented as $\text{Cone } f : \coprod_{i \geq 0} X_i \rightarrow \coprod_{i \geq 0} X_i$, where all $X_i \cong X$; $f_{i,i} = \text{id}_{X_i}$, $f_{i,i+1} = -h$, all other components of f are zero.

It remains to check that any object of \underline{Hw} is a retract of some object of H .

We consider the 'weight resolution' of $X \in \underline{C}^{w=0}$ constructed as in the proof of part III of Theorem 4.3.2 (in fact, it suffices to consider a few last terms). We obtain that the weight complex of X can be presented (in $K_{\mathfrak{w}}(\underline{Hw})$) by $\cdots \rightarrow P_1 \rightarrow P_0$, where $P_i \in \text{Obj} H$. Since it is homotopy equivalent to X , we obtain that X is a retract of P_0 (see part 8 of Proposition 1.3.3). The assertion is proved.

2. By assertion II1, the restriction of representable functors to the category of all coproducts of C_i is a fully faithful functor on \underline{Ht} (see part 4 of Theorem 4.4.2). Since C_i are compact, we can fully faithfully restrict these functors further to C . So it remains to compute the categorical image of this restriction.

Since (\underline{C}, w) admits negative coproducts, \underline{C} contains all coproducts of $C(-, C_i)$. Since C_i are compact, these sums represent functors $\bigoplus C(-, C_i)$ on C . Since \underline{Ht} is abelian, its image also contains all cokernels of morphisms of objects that can be presented as $\bigoplus C(-, C_i)$.

It remains to note that cokernels of morphisms of objects of the type $\bigoplus C(-, C_i)$ give the whole C_* . This fact was mentioned in the Notation, see also Lemma 8.1. of [MVW06]. In fact, this is very easy: every $F : C \rightarrow Ab^{op}$ can be presented as a factor of the natural

$$h : \sum_{i \in I, x \in F(C_i)} C_i \rightarrow F,$$

and the same could be said about the kernel of h .

3. This is just the formula (26). □

Remark 4.5.3. 1. Dualizing part I2, one obtains certain sufficient conditions for right adjoint weight and t -structures to exist. Unfortunately, this requires 'positive' products and cocompact weak cogenerators which do not usually exist (yet see §4.7 of [Bon10]).

2. Since C_i are compact, H could be described as the category of 'formal' coproducts of C_i i.e. $\underline{C}(\prod_{l \in L} C_{i_l}, \prod_{j \in J} C_{i_j}) = \prod_{l \in L} (\bigoplus_{j \in J} \underline{C}(C_{i_l}, C_{i_j}))$; here $i_j, i_l \in I$; I, J are index sets.

3. If \underline{C} is endowed with a t -structure then the question of existence of an adjacent weight structure seems to be difficult in general; cf. Remark 7.1.2 below. Yet see Theorem 4.1 of [Pau08] for an interesting result in this direction (though in rather restrictive conditions).

4.6 The spherical weight structure for the stable homotopy category

We consider the stable homotopy category SH . Recall some of its basic properties.

The objects of SH are called *spectra*. SH contains the *sphere spectrum* S^0 that weakly generates it.

The groups $A_i = SH(S^0[i], S^0)$ are called the stable homotopy groups of spheres. We have $A_i = 0$ for $i < 0$, $A_i = \mathbb{Z}$ for $i = 0$; A_i are finite for $i > 0$. For an arbitrary $A \in Obj SH$ the groups $SH(S^0[i], A)$ are called the homotopy groups of A (they are denoted by $\pi_i(A)$).

The category $SH_{fin} \subset SH$ of finite spectra was defined in Corollary 4.3.3. We will also consider the category $SH_{qfin} \subset SH$ of *quasi-finite spectra*. Its objects are described by the following conditions: all $\pi_i(A)$ are finitely generated and $\pi_i(A) = 0$ for all $i > j$ for some $j \in \mathbb{Z}$. Lastly, we will also mention the full subcategory $SH^- \subset SH$ whose objects are spectra with (stable) homotopy groups that are zero for $i > j$ (for some j that depends on

the spectrum chosen). Obviously, all categories mentioned are triangulated subcategories of SH .

We see that SH_{fin} and SH_{qfin} satisfy the assumptions of part III (version (i)) of Theorem 4.3.2 if we take $H = H'$ equal to the category of finite coproducts of S^0 and $c = \omega$. Indeed, in this case we only need finite sums and their properties which are valid for arbitrary \underline{C} .

Hence we obtain a certain non-degenerate weight structure w on $SH_{fin} \subset SH_{qfin}$. It is bounded above for SH_{qfin} , whereas SH_{fin} is bounded since it is generated by S^0 . Recall that S^0 is compact, hence all objects of H' also are. Hence using part III (version (ii)) of Theorem 4.3.2 we can extend w to SH^- .

Now we describe the heart of w for various categories of spectra. Since $SH(S^0, S^0) = \mathbb{Z}$, we obtain that $H' \cong Ab_{fin.fr}$ (the category of finitely generated free abelian groups); note that $H = H'$ in this case. Since H is idempotent complete, part III (i) of Theorem 4.3.2 implies that $\underline{Hw}_{SH_{fin}} = \underline{Hw}_{SH_{qfin}} \cong Ab_{fin.fr}$.

In SH^- we have $H \cong Ab_{fr}$ (the category of all free abelian groups). Since Ab_{fr} is idempotent complete, we obtain $\underline{Hw}_{SH^-} \cong Ab_{fr}$.

Now recall that SH admits countable (and also, in fact, arbitrary) coproducts. Hence by part I2 of Theorem 4.5.2 we can extend w to the whole SH . This certainly means that $\underline{Hw}_{SH} \cong Ab_{fr}$. Hence the functor t is actually 'strong' for all categories of spectra mentioned, see part 1 of Remark 3.3.4.

Note that any object of $SH^{w=0}$ is isomorphic to a coproduct of spherical spectra. Hence weight Postnikov towers in this case become *cellular towers* for spectra in topology. Their construction and the functoriality properties (in the topological case) are described in §6.3 of [Mar83]; certainly, the results of loc. cit. are parallel to ours. It follows: the corresponding weight spectral sequences (for (co)homological functors defined on SH) are actually Atiyah-Hirzebruch spectral sequences i.e. they relate (co)homology of an arbitrary spectrum X with the cohomology of S^0 (in a way that depends on the cellular tower of X).

Now we describe the relation of the weight complex functor for this weight structure with singular homology and cohomology of spectra.

To this end we recall that SH supports a non-degenerate *Postnikov* t -structure t_{Post} ; the corresponding cohomology functor is given by $SH(S^0, -)$. We obtain that $SH^{-, w \leq 0} = SH^{t_{Post} \leq 0}$. Hence t_{Post} in this case is exactly the t -structure described in part I1 of Theorem 4.5.2. Besides by part 5 of Theorem 4.4.2, any Eilenberg-MacLane spectrum belongs to $SH^{t_{Post}=0}$. Recall that the singular cohomology theory H_{sing}^i for spectra is represented

by the Eilenberg-MacLane spectrum $H\mathbb{Z}$ that corresponds to \mathbb{Z} , whereas the singular homology of X can be calculated as $SH(S^0, H\mathbb{Z} \wedge X[i])$; we will denote it by $H_i^{sing}(X)$ (see the notation of Definition 2.3.1).

We identify $\underline{Hw} = H$ with Ab_{fr} using the functor $H(S^0, -)$; so the target of t is $K(Ab_{fr})$.

Proposition 4.6.1. *Let X be a spectrum.*

1. $H^i(t(X)) \cong H_i^{sing}(X)$.
2. $H^0(Ab(X^{-i}, \mathbb{Z})) \cong H_{sing}^0(X)$.
3. For $X \in Obj SH_{fin}$ we have $X \in SH_{fin}^{w \geq 0} \iff H_{sing}^i(X) = 0 \ \forall i > 0$ and $X \in SH_{fin}^{w \leq 0} \iff H_{sing}^i(X) = 0 \ \forall i < 0$.

Proof. 1. We apply Theorem 2.3.2 to the functor H_0^{sing} . We have $E_1^{pq} = H_q^{sing}(X^p)$ where each X^p is a (possibly, infinite) coproduct of copies of S^0 . Now, the only non-zero homology group of S^0 is \mathbb{Z} placed in dimension 0; the functor $Y \rightarrow H\mathbb{Z} \wedge Y$ commutes with (arbitrary) homotopy colimits and sums.

Hence the spectral sequence $T(H^{sing}, X)$ reduces to the weight complex of X (considered as a complex of free abelian groups). By the convergence condition II(ii) of loc. cit. we have $T(H^{sing}, X) \implies H^{sing}(X)$.

2. Part II3 of Theorem 4.5.2 calculates $SH(X, Y)$ for any Eilenberg-MacLane spectrum Y . In particular, taking $Y = H\mathbb{Z}$ we obtain the claim.

3. As in part 10 of Theorem 4.4.2, if $X \in SH^{w \geq 0}$ (resp. $X \in SH^{w \leq 0}$) then the corresponding conditions on the cohomology of X are fulfilled.

Conversely, let $H_{sing}^i(X) = 0 \ \forall i > 0$. Then by (26) the complex $t(X)$ is acyclic in negative degrees. Since it is a complex of free abelian groups, it is homotopy equivalent to a complex concentrated in non-negative degrees. Hence part IV of Theorem 3.3.1 yields the assertion desired.

The case of X such that $H_{sing}^i(X) = 0$ for all $i < 0$, is considered similarly. \square

Remark 4.6.2. 1. If we take an Eilenberg-MacLane spectrum HI corresponding to some injective group I instead, we will get $SH(X, HI) = Ab(H^0(t(X)), I)$.

2. Note also that $S^{0t \geq 0}$ is exactly $H\mathbb{Z}$. Hence $H\mathbb{Z}$ can be obtained by applying the construction described in the proof of Theorem 4.5.2 to S^0 .

3. The proof of part 1 of Proposition 4.6.1 shows that the weight filtration given by the spherical weight structure on singular homology coincides with the canonical filtration. This is not the case for (stable) homotopy groups of spectra.

5 Idempotent completions; K_0 of categories with bounded weight structures

In §5.1 we recall that an idempotent completion of a triangulated category is triangulated. In §5.2 we prove that a bounded $\underline{\mathcal{C}}$ is idempotent complete (i.e. pseudo-abelian) whenever \underline{Hw} is; in general, the idempotent completion of a bounded $\underline{\mathcal{C}}$ has a weight structure whose heart is the idempotent completion of \underline{Hw} .

In §5.3 we prove: if $\underline{\mathcal{C}}$ is bounded and idempotent complete, then the embedding $\underline{Hw} \rightarrow \underline{\mathcal{C}}$ induces an isomorphism $K_0(\underline{\mathcal{C}}) \cong K_0(\underline{Hw})$. It is a ring isomorphism if $\underline{Hw} \subset \underline{\mathcal{C}}$ are endowed with compatible tensor structures. In §5.4 we study a certain Grothendieck group of endomorphisms in $\underline{\mathcal{C}}$. Unfortunately, it is not always isomorphic to $K_0(\text{End } \underline{Hw})$; yet it is if \underline{Hw} is *regular*; see Definition 5.4.2. Besides, we can still say something about it in other cases. In particular, this allows us to generalize Theorem 3.3 of [BlE07] to arbitrary endomorphisms of motives (in Corollary 5.4.6); see also §8.4 of [Bon09].

In §5.5 we calculate explicitly the groups $K_0(SH_{fin})$ and $K_0(\text{End } SH_{fin})$. It turns out that the classes of $[X]$ and $[g : X \rightarrow X]$ are easily recovered from the rational singular homology of X ; see Proposition 5.5.1. More generally, one can calculate certain groups $K_0(\text{End}^n SH_{fin})$ for $n \in \mathbb{N}$ in a similar way, see Remarks 5.5.2 and 5.4.7.

5.1 Idempotent completions: reminder

We recall that an additive category A is said to be *idempotent complete* (or pseudo-abelian) if for any $X \in \text{Obj } A$ and any idempotent $p \in A(X, X)$ there exists a decomposition $X = Y \oplus Z$ such that $p = i \circ j$, where i is the inclusion $Y \rightarrow Y \oplus Z$, j is the projection $Y \oplus Z \rightarrow Y$.

Any additive A can be canonically idempotent completed. Its idempotent completion is (by definition) the category A' whose objects are (X, p) for $X \in \text{Obj } A$ and $p \in A(X, X) : p^2 = p$; we define

$$A'((X, p), (X', p')) = \{f \in A(X, X') : p'f = fp = f\}.$$

It can be easily checked that this category is additive and idempotent complete, and for any idempotent complete $B \supset A$ we have a natural unique embedding $A' \rightarrow B$.

The main result of [BaS01] (Theorem 1.5) states that an idempotent completion of a triangulated category $\underline{\mathcal{C}}$ has a natural triangulation (with

distinguished triangles being all direct summands of distinguished triangles of \underline{C}).

In this section \underline{C}' will denote the idempotent completion of \underline{C} , \underline{Hw}' will denote the idempotent completion of \underline{Hw} .

Note that if \underline{C} is idempotent complete then \underline{Hw} also is, since $\underline{Hw} \subset \underline{C}$ and is Karoubi-closed in it.

5.2 Idempotent completion of a triangulated category with a weight structure

We prove that \underline{C}^b is idempotent complete if \underline{Hw} is.

Lemma 5.2.1. *If w is bounded, \underline{Hw} is idempotent complete, then \underline{C} also is.*

Proof. We prove that all $\underline{C}^{[i,j]}$ are idempotent complete by induction on $j-i$. The base is: $\underline{C}^{[i,i]} = \underline{C}^{w=0}[-i]$ is idempotent complete.

To make the inductive step it suffices to prove that $\underline{C}^{[-i,1]}$ is idempotent complete if $\underline{C}^{[-i,0]}$ is (for $i > 0$). For $X \in \underline{C}^{[-i,1]}$ and an idempotent $p \in \underline{C}(X, X)$ we consider the functor WD (see part I of Theorem 3.2.2). We obtain an idempotent $q = WD(p) \in K_{\mathbf{w}}^{[0,1]}(\underline{C})(WD(X), WD(X))$ whereas $Y = WD(X)$ has the form $(Z \rightarrow T)$ for $Z, T \in \text{Obj } K_{\mathbf{w}}^b(\underline{C}^{[-i,0]})$. Since $\underline{C}^{[-i,0]}$ is idempotent complete, $K_{\mathbf{w}}^b(\underline{C}^{[-i,0]})$ also is by part 2 of Proposition 3.1.8. Hence there exists a $Z' \rightarrow T'$ and idempotent endomorphisms r, s of Z' and T' , respectively, such that (Y, q) can be presented by the diagram

$$\begin{array}{ccc} Z' & \longrightarrow & T' \\ \downarrow r & & \downarrow s \\ Z' & \longrightarrow & T' \end{array}$$

(in $K_{\mathbf{w}}^{[0,1]}(\underline{C}^{[-i,0]})$).

By Part I5 of Theorem 3.2.2, Z', T' come from a certain weight decomposition of X . Then any corresponding weight decomposition of p is homotopy equivalent to (r, s) . Then part I2 of Theorem 3.2.2 yields that (r, s) also give a weight decomposition of p . Hence the object $(X, p) \in \text{Obj } \underline{C}'$ (see §5.1) can be presented as a cone of a certain map $(Z', r) \rightarrow (T', s)$ in \underline{C}' ; whereas $(Z', r), (T', s) \in \text{Obj } \underline{C}$ by the inductive assumption. □

Now we prove that in the general (bounded) case a weight structure can be extended from \underline{C} to its idempotent completion \underline{C}' .

Proposition 5.2.2. *Let w be bounded. Then the following statements are valid.*

1. *w extends to a bounded weight structure w' for \underline{C}' .*
2. *The heart of \underline{C}' equals $\underline{Hw'}$ (the idempotent completion of \underline{Hw}).*

Proof. 1. By Part II1 of Theorem 4.3.2, we have a bounded weight structure that extends w on the subcategory $D \subset \underline{C}'$ generated by $\underline{Hw'}$. Hence it suffices to recall that D is idempotent complete; see Lemma 5.2.1.

2. Since $\underline{Hw'}$ is idempotent complete, the assertion follows from part II2 of Theorem 4.3.2. □

Corollary 5.2.3. *If (\underline{C}, w) is bounded then $\underline{Hw'}$ generates \underline{C}' .*

Proof. By Proposition 5.2.2, $\underline{Hw'}$ is the heart of a bounded weight structure for \underline{C}' . Now Corollary 1.5.7 yields the result. □

Remark 5.2.4. Possibly the boundedness condition on w in Proposition 5.2.2 can be weakened. However this does not seem to be actual since usually (for the triangulated categories that interest mathematicians) either (\underline{C}, w) is bounded or \underline{C} admits countable coproducts (at least, 'positive' or 'negative' ones). In the latter case \underline{C} is idempotent complete, see Proposition 1.6.8 of [Nee01].

5.3 K_0 of a triangulated category with a bounded weight structure

We recall some standard definitions (cf. 3.2.1 of [GiS96]). We define the Grothendieck group of an additive category A as a group whose generators are of the form $[X]$, $X \in \text{Obj} A$; the relations are $[X \oplus Y] = [X] + [Y]$ for $X, Y \in \text{Obj} A$. The K_0 -group of a triangulated category T is defined as the group whose generators are $[t]$, $t \in \text{Obj} T$; if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle then $[B] = [A] + [C]$. Note that $X \oplus 0 \cong X$ implies that $[X] = [Y]$ if $X \cong Y$ (in A or in T).

For an additive A we define $K_0(K_{\mathfrak{w}}^b(A))$ similarly to $K_0(K^b(A))$; hence it equals $K_0(K^b(A))$ (see Definition 3.1.6).

The existence of a bounded w allows to calculate $K_0(\underline{C})$ easily.

Theorem 5.3.1. *Let (\underline{C}, w) be bounded, let \underline{Hw} be idempotent complete. Then the inclusion $i : \underline{Hw} \rightarrow \underline{C}$ induces an isomorphism $K_0(\underline{Hw}) \rightarrow K_0(\underline{C}^b)$.*

Proof. Since t is an weakly exact functor (see Definition 3.1.6), it gives an abelian group homomorphism $a : K_0(\underline{C}) \rightarrow K_0(K_{\mathfrak{w}}^b(\underline{Hw})) = K_0(K^b(\underline{Hw}))$. By Lemma 3 of 3.2.1 of [GiS96], there is a natural isomorphism $b : K_0(K^b(\underline{Hw})) \rightarrow K_0(\underline{Hw})$. The embedding $\underline{Hw} \rightarrow \underline{C}$ gives a homomorphism $c : K_0(\underline{Hw}) \rightarrow K_0(\underline{C})$. The definitions of a, b, c imply immediately that $b \circ a \circ c = id_{K_0(\underline{Hw})}$. Hence a is surjective, c is injective.

It remains to verify that c is surjective. This follows immediately from the fact that \underline{Hw} generates \underline{C} , see Corollary 1.5.7. □

Remark 5.3.2. Obviously, if \underline{C} is a tensor triangulated category then $K_0(\underline{C})$ is a ring. If the tensor structure on \underline{C} induces a tensor structure on \underline{Hw} then $K_0(\underline{Hw})$ is a ring also and c is a ring isomorphism.

For the convenience of citing we concentrate certain assertions relevant for motives in a single statements.

Proposition 5.3.3. *Suppose that \underline{C} contains an additive negative (see Definition 4.3.1) subcategory H such that H is idempotent complete and \underline{C} is the idempotent completion of $\langle H \rangle$. Then the following statements are valid.*

1. $\langle H \rangle = \underline{C}$.
2. *There exists a conservative weight complex functor $\underline{C} \rightarrow K_{\mathfrak{w}}^b(H)$ which sends $h \in \text{Obj } H$ to $h[0] \in \text{Obj } K_{\mathfrak{w}}^b(H)$. It can be lifted to an exact functor $t^{st} : \underline{C} \rightarrow K^b(H)$ in the case when \underline{C} has a differential graded enhancement (see part 4 of Definition 6.1.2 below).*
3. $K_0(\underline{C}) \cong K_0(H)$.

Proof. 1. By part II of Theorem 4.3.2 there exists a bounded weight structure w' on $\langle H \rangle$ whose heart equals the small envelope of H i.e. to H . Next, by Proposition 5.2.2 w extends to some bounded w on \underline{C} whose heart equals the idempotent completion of H i.e. H . Hence Corollary 5.2.3 immediately yields assertion 1.

2. The weight complex functor $t : \underline{C} \rightarrow K_{\mathfrak{w}}(H)$ can be factorized through $K_{\mathfrak{w}}^b(H)$ since w is bounded. t is conservative by part V of Theorem 3.3.1. If \underline{C} has a differential graded enhancement then t can be lifted to t^{st} by part 2 of Remark 6.2.2 below.

3. Immediate from Theorem 5.3.1. □

5.4 K_0 for categories of endomorphisms

Now we define various Grothendieck groups of endomorphisms in an additive category A . Our definitions are similar to those of [Alm74].

Definition 5.4.1. 1. The generators of $K_0^{add}(\text{End } A)$ are endomorphisms of objects of A ; the relations are: $[g] = [f] + [h]$ if (f, g, h) give an endomorphism of a split short exact sequence.

2. If A is also abelian then we also consider the group $K_0^{ab}(\text{End } A)$. Its generators again are endomorphisms of objects of A ; the relations are: $[g] = [f] + [h]$ if (f, g, h) give an endomorphism of an arbitrary short exact sequence.

3. For a triangulated A we consider the group $K_0^{tr}(\text{End } A)$.

Its generators are endomorphisms of objects of A again; the relations are: $[g] = [f] + [h]$ if (f, g, h) give an endomorphism of a distinguished triangle in A .

Note that $K_0^{ab}(\text{End } A)$ and $K_0^{tr}(\text{End } A)$ are natural factors of $K_0^{add}(\text{End } A)$ (when they are defined). Indeed, $K_0^{ab}(\text{End } A)$ and $K_0^{tr}(\text{End } A)$ have the same generators as $K_0^{add}(\text{End } A)$ and more relations.

Let \underline{C} be bounded. We provide some sufficient conditions for $K_0(\text{End } \underline{C})$ to be isomorphic to $K_0(\text{End } \underline{Hw})$. We need a notion of a *regular* additive category A . Recall that A'_* is the full abelian subcategory of A_* generated by A .

Definition 5.4.2. An additive category A will be called *regular* if it satisfies the following conditions.

1. A is isomorphic to its small envelope (see part 3 of Definition 4.3.1) i.e. if $X, Y \in \text{Obj } A$, X is a retract of Y , is then X has a complement to Y (in A).

2. Every object of A'_* has a finite resolution by objects of A .

The most simple examples of regular categories are abelian semisimple categories and the category of finitely generated projective modules over a Noetherian (commutative) local ring all of whose localizations are regular local; cf. the end of §1 of [Alm74].

We will need the following technical statement. Let R be an associative ring with a unit.

Lemma 5.4.3. 1. If A is regular then $K_0^{add}(\text{End } A) \cong K_0^{ab}(\text{End } A'_*)$.

2. If A is the category of finitely generated projective modules over R then A'_* is the category of all (left) modules over R .

Proof. 1. We apply the method of the proof of Proposition 5.2 of [Alm73]. First we consider the obvious category $\text{End } \underline{Hw}'_*$ and note that it is abelian. Next, the objects of \underline{Hw} become projective in \underline{Hw}'_* . Hence all 3-term complexes in \underline{Hw} that become exact in \underline{Hw}'_* do split in \underline{Hw} . Therefore we can

define $K_0^{add}(\text{End } \underline{Hw}_*)$ as the Grothendieck group of an exact subcategory of $\text{End } \underline{Hw}'_*$.

Condition 1 of Definition 5.4.2 ensures that for any short exact sequence $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ in $\text{End } \underline{Hw}'_*$ if $G, G'' \in \text{End } \underline{Hw}$ then $G' \in \text{End } \underline{Hw}$ (i.e. G' is an endomorphism of an object of \underline{Hw}). Lastly, condition 2 of Definition 5.4.2 easily implies that any $G \in \text{End } \underline{Hw}'_*$ has a finite resolution by objects of $\text{End } \underline{Hw}$ (again note that objects of \underline{Hw} become projective in \underline{Hw}'_*). Hence applying Theorem 16.12 of [Swa68] (page 235) we obtain the result.

2. The equivalence is given by sending a functor F to $F(R)$ (here R is considered as right R -module: this endows $F(R)$ with a left R -module structure); and an R -module Q to $P \mapsto \text{Hom}_R(P, Q)$. Note that all $F(P)$ can be uniquely recovered from $F(R)$ since all finitely generated projective modules are direct summands of R^m (for some $m > 0$).

□

Proposition 5.4.4. 1. *There exist natural homomorphisms $K_0(\text{End } \underline{Hw}) \xrightarrow{c} K_0^{tr}(\text{End } \underline{C}) \xrightarrow{d} K_0^{ab}(\text{End } \underline{Hw}'_*)$; c is a surjection.*

2. *c is an isomorphism if \underline{Hw} is regular.*

Proof. 1. c is induced by $i : \underline{Hw} \rightarrow \underline{C}$. For $g : X \rightarrow X$ we define

$$d(g) = \sum (-1)^i [g_{i*} : H^i(t(X)) \rightarrow H^i(t(X))]. \quad (28)$$

Here $H^i(t(X)) \in \text{Obj } \underline{Hw}'_*$ are the cohomology of the weight complex; see part 2 of Remark 3.1.7. We obtain a well-defined homomorphism since t is a weakly exact functor (see Definition 3.1.6); see part 3 of Remark 3.1.7.

c is surjective since for $g : X \rightarrow X$ we have the equality $[g] = \sum (-1)^i [g^i : X^i \rightarrow X^i]$. This equality follows easily from the fact that a repetitive application of the (single, shifted) weight decomposition functor to a morphism yields its infinite weight decomposition (see Theorem 3.2.2; note that X is bounded).

2. In the case when \underline{Hw} is abelian semi-simple we have $\underline{Hw} = \underline{Hw}'_*$. Hence the equality $d \circ c = \text{id}_{K_0(\text{End } \underline{C})}$ yields the assertion (in this case).

Now, in the general (regular) case it suffices to apply the equality $K_0(\text{End } \underline{C}) = K_0(\text{End } \underline{Hw}'_*)$ (this is part 1 of Lemma 5.4.3).

□

Remark 5.4.5. 1. Unfortunately, c is not an isomorphism in the general case. To see this it suffices to consider the example described in part 3 of Remark 1.5.2 for $\underline{C} = K^b(Z)$ where Z is the category of free $\mathbb{Z}/4\mathbb{Z}$ -modules. This

fact is also related to the observation in the end of §1 of [Alm74]. Certainly, Z is not regular.

2. Certainly, if $i : \underline{Hw} \rightarrow \underline{C}$ is a tensor functor then c, d are ring homomorphisms, cf. Remark 5.3.2.

The surjectivity of c immediately implies the following fact.

Corollary 5.4.6. *Let $r : \underline{C} \rightarrow D^b(R)$ and $s : \underline{C} \rightarrow D^b(S)$ be exact functors for abelian R, S ; let $r_* : K_0(\underline{C}) \rightarrow K_0(D^b(R))$ and $s_* : K_0(\underline{C}) \rightarrow K_0(D^b(S))$ be the induced homomorphisms. Let $u : K_0(\text{End } D^b(R)) \rightarrow K_0(\text{End } R)$ and $v : K_0(\text{End } D^b(S)) \rightarrow K_0(\text{End } S)$ be defined as $(g : X \rightarrow X) \rightarrow [g_{i*} : H^i(X) \rightarrow H^i(X)]$. Let T be an abelian group; $x : K_0(\text{End } R) \rightarrow T$ and $y : K_0(\text{End } S) \rightarrow T$ be group homomorphisms. Then the equality $x \circ u \circ r_* \circ c = y \circ v \circ s_* \circ c$ implies $x \circ u \circ r_* = y \circ v \circ s_*$.*

In particular, one can take $\underline{C} = DM_{gm}^{eff} \mathbb{Q}$, $\underline{Hw} = \text{Chow}^{eff} \mathbb{Q}$ (see Remark 6.6.1 below), r, s given by l -adic cohomology realizations (for two distinct l 's), and x, y given by traces of endomorphisms. It follows that the alternated sum of traces of maps induced by a $g \in DM_{gm}^{eff} \mathbb{Q}(X, X)$ on the cohomology of X does not depend on l . We also obtain the independence from l of $n_\lambda(H) = (-1)^i n_{\lambda} g_{H^i(X)}^*$; here $n_{\lambda} g_{H^i(X)}^*$ for a fixed algebraic λ denotes the algebraic multiplicity of the eigenvalue λ for the operator $g_{H^i(X)}^*$.

This generalizes Theorem 3.3 of [BlE07] to arbitrary morphisms of motives; see §8.4 of [Bon09] for more details.

Lastly, we consider some more general K_0 -groups.

Remark 5.4.7. 1. For an additive A instead of $\text{End } A$ one can for any $n \geq 0$ consider the category $\text{End}^n A$ whose objects are the following $n + 1$ -tuples: $(X \in \text{Obj } A; g_1, \dots, g_n \in A(X, X))$. We have $\text{End}^0(A) = A$, $\text{End}^1(A) = \text{End } A$. Generalizing Definition 5.4.1 in an obvious way one defines $K_0^{add}(\text{End}^n A)$, $K_0^{ab}(\text{End}^n A)$, and $K_0^{tr}(\text{End}^n A)$ (for A additive, abelian or triangulated, respectively). Next, one can define c, d as in Proposition 5.4.4; exactly the same argument as in the proof of the Proposition shows that c is always surjective and it is also injective if \underline{Hw} is regular. In particular, this is true for $\underline{C} = SH_{fin}$; see Proposition 5.5.1 below.

2. Even more generally, for any ring R one could consider the category $\text{End}(R, A)$ of R -representations in A i.e. of pairs $(X, H : R \rightarrow A(X, X))$; here $X \in \text{Obj } A$, H is a unital homomorphism of rings. In particular, we have $\text{End}(R, A) = A$ for $R = \mathbb{Z}$, $= \text{End } A$ for $R = \mathbb{Z}[t]$, and $= \text{End}^n A$ for $R = \mathbb{Z}\langle t_1, \dots, t_n \rangle$ (the algebra of non-commutative polynomials). Again one defines $K_0^{add}(\text{End}(R, A))$, $K_0^{ab}(\text{End}(R, A))$, and $K_0^{tr}(\text{End}(R, A))$, c and d . Yet the method of the proof of Proposition 5.4.4 fails for a general R ; one can only note that $d \circ c$ is an isomorphism if \underline{Hw} is abelian semi-simple.

5.5 An application: calculation of $K_0(SH_{fin})$ and $K_0(End SH_{fin})$

Now we calculate explicitly the groups $K_0(SH_{fin})$ and $K_0^{tr}(End SH_{fin})$. The author doesn't think that (all of) these results are new; yet they illustrate our methods very well.

We will need the following simple observation: $K_0(A)$ is naturally a direct summand of $K_0(End A)$ (both in the 'triangulated' and in the 'additive' case). The splitting is induced by $[f : X \rightarrow X] \rightarrow [X] \rightarrow [0 : X \rightarrow X]$; see §1 of [Alm74].

We define the group Λ as a subgroup of the multiplicative group $\Lambda(\mathbb{Z}) = \{1 + t\mathbb{Z}[[t]]\}$ that is generated by polynomials (with constant term 1). Λ and $\Lambda(\mathbb{Z})$ are also rings; see Proposition 3.4 of [Alm73] for Λ and [Haz78] for $\Lambda(\mathbb{Z})$.

Proposition 5.5.1. *1. $K_0(SH_{fin}) \cong \mathbb{Z}$ with the isomorphism sending $X \in Obj SH_{fin}$ to $[X] = \sum (-1)^i \dim_{\mathbb{Q}}(H_i^{sing}(X) \otimes \mathbb{Q})$ (the rational singular homology of X).*

2. $K_0^{tr}(End SH_{fin}) \cong \mathbb{Z} \oplus \Lambda$ with the isomorphism sending $g : X \rightarrow X$ to $[X] \oplus \prod_i (\det_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q}))^{(-1)^i}$; here $g_i t \otimes \mathbb{Q}$ is the map induced by $g \otimes t$ on $H_i^{sing}(X) \otimes_{\mathbb{Z}} \mathbb{Q}[t]$.

Proof. 1. We have $\underline{Hw} = Ab_{fin.fr}$ for the spherical weight structure w on SH_{fin} ; see §4.6. Hence $K_0(SH_{fin}) \cong K_0(Ab_{fin.fr}) = K_0(\mathbb{Z}) = \mathbb{Z}$.

The second assertion could easily be deduced from part 1 of Proposition 4.6.1. Note that $K_0(SH_{fin})$ is a direct summand of $K_0^{tr}(End SH_{fin})$; hence

$$[X] = \sum (-1)^i [H^i(t(X))] = \sum (-1)^i [H_i^{sing}(X)]$$

by (28). We also use the fact that $K_0(\mathbb{Z})$ injects into $K_0(\mathbb{Q})$, so $[H_i^{sing}(X)]$ can be computed rationally.

2. By part 2 of Lemma 5.4.3 we have $\underline{Hw}_* \cong Ab_{fin.fr}$ (the category of finitely generated abelian groups). Hence \underline{Hw} is regular (see Definition 5.4.2). Therefore by part 2 of Proposition 5.4.1 we have $K_0^{tr}(End SH_{fin}) \cong K_0^{add}(End Ab_{fin.fr})$. Then the Main Theorem in §1 of [Alm74] implies that $K_0^{tr}(End SH_{fin}) \cong \mathbb{Z} \oplus \Lambda$.

Next, (28) implies $[g] = (-1)^i [g_{i*}]$. Now note that $\Lambda(\mathbb{Q}) \rightarrow \Lambda(\mathbb{Z})$ is injective; so it suffices to calculate $[g_{i*}]$ rationally. Lastly, the equality

$$[g_{i*} \otimes \mathbb{Q}] = \dim_{\mathbb{Q}}(H_i^{sing}(X) \otimes \mathbb{Q}) \oplus \det_{\mathbb{Q}[t]}(id - g_i t \otimes \mathbb{Q})$$

follows from the formula at the bottom of p. 376 of [Alm74]. □

Remark 5.5.2. 1. Note that the isomorphisms described are compatible with the natural ring structures of K_0 -groups involved.

2. Assertion 1 doesn't seem to be new; yet the author doesn't know of any paper that contains assertion 2 in its current form.

3. One also has $K_0^{tr}(\text{End}^n SH_{fin}) \cong K_0^{add}(\text{End}^n Ab_{fin.fr})$; see Remark 5.4.7.

6 Twisted complexes over negative differential graded categories; Voevodsky's motives

The goal of this section is to apply our theory to triangulated categories that have differential graded enhancements (as considered in [Bon09]); this will allow to use it for motives.

In §6.1 we recall the definitions of differential graded categories and twisted complexes over them. In §6.2 we consider negative differential graded categories; we obtain a weight structure on the category of twisted complexes (over them). In §6.3 we construct the *truncation functors* t_N ; t_0 is the *strong weight complex functor* for this case (see Conjecture 3.3.3).

In §6.4 we recall the spectral sequence $S(H, X)$ constructed in §7 of [Bon09] for H having a differential graded enhancement, and prove that it can be obtained from $T(H, X)$ by means of a certain shift of indices. In particular, this shows that S does not depend on the choice of enhancements. We also prove that *truncated realizations* for representable realizations are represented by the adjacent t -truncations of representing objects (see also §2.5 and §7.1).

In §6.5 we apply our theory to Voevodsky's motivic categories $DM_{gm}^{eff} \subset DM_{gm}$.

We calculate the heart of the *Chow* weight structures obtained in §6.6. We also recall that we can apply this theory with rational coefficients over a perfect field k of arbitrary characteristic.

6.1 Basic definitions

We recall relevant definitions for differential graded categories as they were presented in [Bon09]; cf. also [BeV08] and [BoK90].

Categories of *twisted complexes* were first considered in [BoK90]. However our notation differs slightly from those of [BoK90]; some of the signs are also different.

An additive category C is called *graded* if for any $P, Q \in \text{Obj } C$ there is a canonical decomposition $C(P, Q) \cong \bigoplus_i C^i(P, Q)$ defined; this decomposition

should satisfy $C^i(*, *) \circ C^j(*, *) \subset C^{i+j}(*, *)$. A differential graded category (cf. [BoK90] or [BeV08]) is a graded category endowed with an additive operator $\delta : C^i(P, Q) \rightarrow C^{i+1}(P, Q)$ for all $i \in \mathbb{Z}, P, Q \in \text{Obj} C$. δ should satisfy the equalities $\delta^2 = 0$ (so $C(P, Q)$ is a complex of abelian groups); $\delta(f \circ g) = \delta f \circ g + (-1)^i f \circ \delta g$ for any $P, Q, R \in \text{Obj} C, f \in C^i(P, Q), g \in C(Q, R)$. In particular, $\delta(id_P) = 0$.

We denote δ restricted to morphisms of degree i by δ^i .

For any additive category A one can construct the following differential graded categories.

We denote the first one by $S(A)$. We take $\text{Obj} S(A) = \text{Obj} A$; $S(A)^i(P, Q) = A(P, Q)$ for $i = 0$; $S(A)^i(P, Q) = 0$ for $i \neq 0$. We take $\delta = 0$.

We also consider the category $B^b(A)$ whose objects are the same as for $C^b(A)$ whereas for $P = (P^i), Q = (Q^i)$ we define $B^b(A)^i(P, Q) = \bigoplus_{j \in \mathbb{Z}} A(P^j, Q^{i+j})$. Obviously $B^b(A)$ is a graded category. $B(A)$ will denote the unbounded analogue of $B^b(A)$.

We take $\delta f = d_Q \circ f - (-1)^i f \circ d_P$, where $f \in B^i(P, Q)$, d_P and d_Q are the differentials in P and Q . Note that the kernel of $\delta^0(P, Q)$ coincides with $C(A)(P, Q)$ (the morphisms of complexes); the image of δ^{-1} are the morphisms homotopic to 0.

$B^b(A)$ can be obtained from $S(A)$ by means of the category functor Pre-Tr described below.

For any differential graded C we define a category $H(C)$; its objects are the same as for C ; its morphisms are defined as

$$H(C)(P, Q) = \text{Ker } \delta_C^0(P, Q) / \text{Im } \delta_C^{-1}(P, Q).$$

Having a differential graded category C one can construct another differential graded category Pre-Tr(C) as well as a triangulated category Tr(C). The simplest example of these constructions is Pre-Tr($S(A)$) = $B^b(A)$.

Definition 6.1.1. The objects of Pre-Tr(C) are

$$\{(P^i), P^i \in \text{Obj} C, i \in \mathbb{Z}, q_{ij} \in C^{i-j+1}(P^i, P^j)\};$$

here almost all P^i are 0; for any $i, j \in \mathbb{Z}$ we have

$$\delta q_{ij} + \sum_l q_{lj} \circ q_{il} = 0 \tag{29}$$

We call q_{ij} *arrows* of degree $i - j + 1$. For $P = \{(P^i), q_{ij}\}, P' = \{(P'^i), q'_{ij}\}$ we set

$$\text{Pre-Tr}_l(P, P') = \bigoplus_{i, j \in \mathbb{Z}} C^{l+i-j}(P^i, P'^j).$$

For $f \in C^{l+i-j}(P^i, P'^j)$ (an arrow of degree $l+i-j$) we define the coboundary of the corresponding morphism in $\text{Pre-Tr}(C)$ as

$$\delta_{\text{Pre-Tr}(C)}f = \delta_C f + \sum_m (q'_{jm} \circ f - (-1)^{(i-m)l} f \circ q_{mi}).$$

It can be easily seen that $\text{Pre-Tr}(C)$ is a differential graded category (see [BoK90]). There is also an obvious translation functor on $\text{Pre-Tr}(C)$. Note also that the terms of the complex $\text{Pre-Tr}(C)(P, P')$ do not depend on q_{ij} and q'_{ij} whereas the differentials certainly do.

We denote by $Q[j]$ the object of $\text{Pre-Tr}(C)$ that is obtained by putting $P^i = Q$ for $i = -j$, all other $P^j = 0$, all $q_{ij} = 0$. We will write $[Q]$ instead of $Q[0]$.

Immediately from the definition we have $\text{Pre-Tr}(S(A)) \cong B^b(A)$.

A morphism $h \in \text{Ker } \delta^0$ (a closed morphism of degree 0) is called a *twisted morphism*. For a twisted morphism $h = (h_{ij}) \in \text{Pre-Tr}((P^i, q_{ij}), (P'^i, q'_{ij}))$, $h_{ij} \in C(P^i, P'^j)$ we define $\text{Cone}(h) = (P''^i, q''_{ij})$, where $P''^i = P^{i+1} \oplus P^i$,

$$q''_{ij} = \begin{pmatrix} q_{i+1,j+1} & 0 \\ h_{i+1,j} & q'_{ij} \end{pmatrix}$$

We have a natural triangle of twisted morphisms

$$P \xrightarrow{f} P' \rightarrow \text{Cone}(f) \rightarrow P[1], \quad (30)$$

the components of the second map are $(0, id_{P^i})$ for $i = j$ and 0 otherwise. This triangle induces a triangle in the category $H(\text{Pre-Tr}(C))$.

Now we define distinguished triangles in $\text{Tr}(C)$ and a certain differential graded subcategory $\text{Pre-Tr}^+(C) \subset \text{Pre-Tr}(C)$.

Definition 6.1.2. 1. For distinguished triangles in $\text{Tr}(C)$ we take the triangles isomorphic to those that come from (30) for $P, P' \in \text{Pre-Tr}(C)$.

2. $\text{Pre-Tr}^+(C)$ is defined as a full subcategory of $\text{Pre-Tr}(C)$. $\{(P^i), q_{ij}\} \in \text{Obj } \text{Pre-Tr}^+(C)$ if there exist $m_i \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ we have $q_{ij} = 0$ for $i + m_i \geq j + m_j$.

3. $\text{Tr}^+(C)$ is defined as $H(\text{Pre-Tr}^+(C))$; the definition of distinguished triangles is the same as for $\text{Tr}(C)$.

4. We will say that \underline{C} admits a *differential graded enhancement* if it is equivalent to $\text{Tr}(C)$ for some differential graded C .

We summarize the properties of categories defined that are most relevant for the current paper; see [Bon09] and [BoK90] for the proofs.

Proposition 6.1.3. *$ITr^+(C) \subset Tr(C)$ (the embedding is full) are triangulated categories.*

II For any additive category A there are natural isomorphisms

1. $\text{Pre-Tr}(B(A)) \cong B(A)$.

2. $Tr(B(A)) \cong K(A)$.

3. $Tr(S(A)) \cong K^b(A)$

III 1. There are natural embeddings of categories $i : C \rightarrow \text{Pre-Tr}^+(C)$ and $H(C) \rightarrow Tr^+(C)$ sending P to $[P]$.

2. Pre-Tr , Tr , Pre-Tr^+ , Tr^+ are functors on the category of differential graded categories i.e. any differential graded functor $F : C \rightarrow C'$ naturally induces functors $\text{Pre-Tr}F$, TrF , $\text{Pre-Tr}^+ F$, and $Tr^+ F$.

4. Let $F : \text{Pre-Tr}^+(C) \rightarrow D$ be a differential graded functor. Then the restriction of F to $C \subset \text{Pre-Tr}(C)$ gives a differential graded functor $FC : C \rightarrow D$. Moreover, since $FC = F \circ i$, we have $\text{Pre-Tr}^+(FC) = \text{Pre-Tr}^+(F) \circ \text{Pre-Tr}^+(i)$; therefore $\text{Pre-Tr}^+(FC) \cong \text{Pre-Tr}^+(F)$.

IV $Tr^+(C)$ as a triangulated category is generated by the image of the natural map $\text{Obj}C \rightarrow \text{Obj}Tr^+(C) : P \rightarrow [P]$.

For example, for $X = (P^i, q_{ij}) \in \text{Obj Pre-Tr}(C)$ we have $\text{Pre-Tr}F(X) = (F(P^i), F(q_{ij}))$; for a morphism $h = (h_{ij})$ of $\text{Pre-Tr}(C)$ we have $\text{Pre-Tr}F(h) = (F(h_{ij}))$. Note that the definition of $\text{Pre-Tr}F$ on morphisms does not involve q_{ij} ; yet $\text{Pre-Tr}F$ certainly respects differentials for morphisms.

6.2 Negative differential graded categories; a weight structure for $Tr(C)$

Suppose now that a differential graded category C is *negative* i.e. for any $X, Y \in \text{Obj}C$ we have: $i > 0 \implies C^i(X, Y) = \{0\}$ (cf. Definition 4.3.1).

For $\underline{C} = Tr^+(C)$ (in this case it also equals $Tr(C)$) we define $\underline{C}^{w \leq 0}$ as a class that contains all objects that are isomorphic to those that satisfy $P^i = 0$ for $i > 0$. $\underline{C}^{w \geq 0}$ is defined similarly by the condition $P^i = 0$ for $i < 0$.

Proposition 6.2.1. *1. $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ yield a bounded weight structure for \underline{C} .*

2. \underline{Hw} is isomorphic to the small envelope of HC (cf. Definition 4.3.1).

Proof. The definition of morphisms in \underline{C} immediately yields that $\underline{C}^{w \geq 0} \perp \underline{C}^{w \leq 0}[1]$. We obviously have $\underline{C}^{w \leq 0}[1] \subset \underline{C}^{w \leq 0}$; $\underline{C}^{w \geq 0} \subset \underline{C}^{w \geq 0}[1]$. The verification of the fact that $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ are Karoubi-closed in \underline{C} is straightforward. However we will never actually use this statement below (so we can replace $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ described by their Karoubizations in the definition of w).

It remains to check that any object X of \underline{C} admits a weight decomposition. We follow the proof of Proposition 2.6.1 of [Bon09].

We take $(P^i, f_{ij}, i, j \leq 0)$ as $X^{w \leq 0}$ and $(P^i, f_{ij}, i, j \geq 1)[1]$ as $X^{w \geq 1}$. We should verify that $X^{w \leq 0}$ and $X^{w \geq 1}$ are objects of \underline{C} .

We have to check that the equality (29) is valid for $X^{w \leq 0}$ (resp. $X^{w \geq 1}$). All terms of (29) are zero unless $i \leq j \leq 0$ (resp. $1 \leq i \leq j$). Moreover, in the case $i \leq j \leq 0$ (resp. $1 \leq i \leq j$) the terms of (29) are the same as for X . Both of these facts follow immediately from the negativity of C .

Now we verify that $(id_{P^i}, i \leq 0)$ gives a morphism $X \rightarrow X^{w \leq 0}$ and $(id_{P^i}, i \leq 1)$ gives a morphism $X^{w \geq 1}[-1] \rightarrow X$. Indeed, for these morphisms the equality $\delta_{\text{Pre-Tr}(C)} f = 0$ is obvious by the negativity of C .

Next we should check that $X \rightarrow X^{w \leq 0}$ is the second morphism of the triangle corresponding to $X^{w \leq 1}[1] \rightarrow X$; this easily follows from (30).

2. Obviously, the objects of HC belong to $\underline{C}^{w=0}$. Next, the definition of \underline{C} easily yields that $\underline{Hw}(X, Y) \cong HC(X, Y)$ for $X, Y \in \text{Obj } HC$.

Moreover, assertion 1 implies that any object of \underline{C} has a 'filtration' by subobjects whose 'successive factors' come from HC . By part II2 of Theorem 4.3.2 we obtain that \underline{Hw} is isomorphic to the small envelope of HC . \square

Obviously, the same construction also gives weight structures for all unbounded versions of $\text{Tr}(C)$.

Remark 6.2.2. 1. Alternatively, Proposition 6.2.1 could be deduced from part II of Theorem 4.3.2. In particular, this method easily allows to prove that $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ are Karoubi-closed using the fact that the small envelope of HC lies in both of them (cf. the beginning of the proof of Part II2 loc. cit.).

2. Let $\underline{C} = \text{Tr}^+(C)$ for C such that $H^i C(X, Y) = 0$ for all $i > 0$, $X, Y \in \text{Obj } C$ (the cohomology of $C(-, -)$ is concentrated in non-positive degrees). Let C_- be a (non-full!) subcategory of C with the same objects and $C_-(X, Y) = C(X, Y)^{t \leq 0}$ (morphisms are the zeroth canonical truncation of those of C). Then by part 2 of Remark 2.7.4 of [Bon09] the embedding $C_- \rightarrow C$ induces an equivalence of triangulated categories $\text{Tr}^+(C_-) \rightarrow \text{Tr}^+(C)$.

It follows: if $\underline{C} \cong \text{Tr}^+(C)$ and $C(-, -)$ is acyclic in positive degrees then we can assume C to be negative. In particular, the strong (i.e. exact) weight complex functor $\underline{C} \rightarrow K^b(\underline{Hw})$ exists in this case (see below).

6.3 Truncation functors; comparison of weight complexes

For $N \geq 0$, $P, Q \in \text{Obj } C$ we denote the $-N$ -th canonical filtration of $C(P, Q)$ (i.e. $C^{-N}(P, Q)/d_P C^{-N-1}(P, Q) \rightarrow C^{-N+1}(P, Q) \rightarrow \dots \rightarrow C^0(P, Q) \rightarrow 0$) by $C_N(P, Q)$.

We denote by C_N the differential graded category whose objects are the same as for C whereas the morphisms are given by $C_N^i(P, Q)$. The composition of morphisms is induced by those in C . For morphisms in C_N presented by $g \in C^i(P, Q)$, $h \in C^j(Q, R)$, we define their composition as the morphism represented by $h \circ g$ for $i + j \geq -N$ and zero for $i + j < -N$. Certainly, all C_N are negative (i.e. all morphisms of positive degrees are zero).

We have an obvious functor $C \rightarrow C_N$. By part III2 of Proposition 6.1.3, this gives canonically a functor $t_N : \underline{C} \rightarrow Tr(C_N)$. We denote $Tr(C_N)$ by \underline{C}_N .

Obviously, objects of \underline{C}_N can be represented as certain $(P^i, f_{ij} \in C_N^{i-j+1}(P^i, P^j), i < j \leq i + N + 1)$, the morphisms between (P^i, f_{ij}) and (P^i, f'_{ij}) are represented by certain $g_{ij} \in C_N^{i-j}(P^i, P^j)$, $i \leq j \leq i + N$, etc. The functor t_N 'forgets' all elements of $C^m([P], [Q])$ for $P, Q \in Obj C$, $m < -N$, and factorizes $C^{-N}([P], [Q])$ modulo coboundaries. In particular, for $N = 0$ we get ordinary complexes over HC i.e. $\underline{C}_0 = K^b(HC)$.

t_0 will be called the *strong weight complex* functor.

One could easily verify that the strong weight complex functor constructed is a lift of the weight complex functor t corresponding to the weight structure w to an exact functor t^{st} (as in Conjecture 3.3.3). This follows immediately from the explicit description of $X^{w \leq 0}$ and $X^{w \geq 1}$ for any $X \in Obj \underline{C}$ (in the proof of Proposition 6.2.1).

Conjecture 6.3.1. 1. For a general (\underline{C}, w) there also exist certain exact higher truncation functors t_N such that t_0 is the 'strong' weight complex functor; cf. Conjecture 3.3.3. Their targets \underline{C}_N should satisfy: if $X, Y \in \underline{C}^{w=0}$ then $\underline{C}_N(t_N(X), t_N(Y)[-i]) = \underline{C}(X, Y)$ for $0 \leq i \leq N$ and $= \{0\}$ otherwise. These categories should admit full embeddings $i_N : \underline{C}^{[0, N]} \rightarrow \underline{C}_N$; distinguished triangles of \underline{C} consisting of elements of $\underline{C}^{[0, N]}$ should be mapped to distinguished triangles by i_N .

2. Let $I : \underline{C} \rightarrow D(A)$ be an exact functor, where \underline{C}, w is a triangulated category with a weight structure, A is an abelian category. If $I(\underline{C}^{w=0}) \subset D_{[0, N]}(A)$ (i.e. acyclic for degrees outside $[0, N]$) then I can be factorized through t_N .

6.4 Weight spectral sequences for enhanced realizations

The method of construction of weight spectral sequences in [Bon09] was somewhat distinct from the method we use here. In [Bon09] we used a certain filtration on the complex that computes cohomology; that filtration can be obtained from the filtration corresponding to our current method by Deligne's decalage (see §1.3 of [Del71] or [Par96]). So the spectral sequence there was

'shifted one level down' (in particular, it was functorial in X starting from E_1 already). We compare the methods here.

Let J be some negative differential graded category, let $\mathfrak{H} = Tr(J)$, $J' = \text{Pre-Tr}(J)$. Below we will use the same notation for Voevodsky's motives (which are the most important example of this situation).

In [Bon09] weights were constructed only for (co)homological functors that admit an enhancement i.e. those that can be factorized through $Tr(F)$ for a differential graded functor $F : J \rightarrow C$. Here we consider only $C = B(A)$ for an abelian A and homological functors of the form $H_{K_A} \circ Tr(F)$ (here H_{K_A} denotes the zeroth cohomology functor for $C(A)$). The cohomological functor case (also with $C = B(A)$) was considered in §7.3 of [Bon09] (certainly, reversing the arrows is no problem). Note still that for those realizations for which $C \neq B(A)$ one can sometimes reduce the situation to the case $C = B(A')$ for a large A' (in particular, this seems to be the case for the Hodge realization).

Now we recall the formalism of [Bon09] (modified for the homological functor case).

We denote the functor $\text{Pre-Tr}(F) : J' \rightarrow B(A)$ by G , denote $Tr(F) : \mathfrak{H} \rightarrow K(A)$ by E . It turns out that the virtual t -truncations of F (see Remark 2.5.2) have nice differential graded enhancements.

We recall that for a complex Z over A , $b \in \mathbb{Z}$, its b -th canonical truncation from above is the complex $\cdots \rightarrow Z^{b-1} \rightarrow \text{Ker}(Z^b \rightarrow Z^{b+1})$, here $\text{Ker}(Z^b \rightarrow Z^{b+1})$ is put in degree b .

For any $b \geq a \in \mathbb{Z}$ we consider the following functors. By $F_{\tau_{\leq b}}$ we denote the functor that sends $[P]$ to $\tau_{\leq b}(F([P]))$. These functors are differential graded; hence they extend to $G_i = \text{Pre-Tr}(F_{\tau_{\leq -i}}) : J' \rightarrow B(A)$. Note that we consider the $-i$ -th filtration here in order to make the filtration decreasing (which is usual when the decalage is applied); this is another minor distinction of the current exposition from those of [Bon09]. The functors $Tr(F_{\tau_{\leq -i}})$ were called truncated realizations also in loc.cit.

Let $X = (P^i, q_{ij}) \in \text{Obj } J'$. The complexes $G_b(X)$ give a filtration of $G(X)$; one could also consider $G_{a,b}(X) = G_b(X)/G_{a-1}(X)$. We obtain the spectral sequence of a filtered complex (see §III.7.5 of [GeM03])

$$S : E_1^{ij}(S) \implies H^{i+j}(G(X)). \quad (31)$$

Here $E_1^{ij}(S) = H^{i+j}(G_{1-j}(X)/G_{-j}(X))$.

All $G_b(X)$ are J' -contravariantly functorial with respect to X . Besides, starting from E_1 the terms of S depend only on the homotopy classes of $G_b(X)$. Hence starting from E_1 the terms of S are functorial with respect to X (considered as an object of \mathfrak{H}).

Now we compare spectral sequences obtained using this method with those provided by Theorem 2.4.2. In fact, the comparison statement could be proved by considering the derived exact couple for T ; see (part 3 of) Remark 2.5.1. Alternatively, we could have extended Theorem 4.4.2; see parts 5 and 4 of Remark 4.4.3. Instead, here we give a proof in terms of filtered complexes.

To this end we compare the filtrations of $G(X)$ corresponding to T and S . Fortunately, we don't have to write down the differential in G ; it suffices to recall that $G^j(X) = \bigoplus_{k+l=j} F_k(P^l)$.

The method of Theorem 2.3.2 gives the following filtration on $G(X)$: $Q_i G^j(X) = \bigoplus_{k+l=j, l \geq i} F_k(P^l)$.

Now we apply decalage to this filtration. It is easily seen that we obtain the filtration given by G_i i.e.

$$(DecQ)_i(G^j(X)) = \bigoplus_{k+l=j, l \geq j+i+1} F_k(P^l) \bigoplus \text{Ker}(F_{-b}(P^{l+i}) \rightarrow F_{-b+1}(P^{l+i})).$$

Hence $T_{n+1}^{pq} = S_n^{-q, p+2q}$ for all integral i, j and $n > 0$; the corresponding filtrations on the limit (i.e. on $H^{i+j}(E(X))$) coincide up to a certain shift of indices.

In §7.3 of [Bon09] so-called *truncated realizations* were considered. They were defined as $Tr(F_{\tau \leq b})$ and $Tr(F_{\tau \leq b}/F_{\tau \leq a-1}) : \mathfrak{H} \rightarrow K(A)$ (for $a \leq b \in \mathbb{Z}$). The formula (15) of [Bon09] computes all $E_n^{ij}(S)$ for $n \geq 1$ in terms of the weight filtration of truncated realizations of X ; this description is \mathfrak{H} -functorial.

Remark 6.4.1. 1. Suppose now that there exists a differential graded functor $F^1 : J \rightarrow B(A)$ and a differential graded transformation $F^1 \rightarrow F$ such that the induced cohomology functor maps are isomorphisms in degrees $\leq b$ and are zero in degree $> b$. Let F^2 denote $F_{\tau \leq b}^1$. We have a natural transformation $F^2 \rightarrow F^1$ which is an isomorphism on cohomology. Hence by part 2 of Corollary 2.7.2 of [Bon09], the transformation of functors $Tr(F^2 \rightarrow F^1)$ induces quasi-isomorphisms of their values. Next, the transformation $F^1 \rightarrow F$ induces a transformation $F^2 \rightarrow F_{\tau \leq b}$. Applying part 2 of Corollary 2.7.2 of [Bon09] again we obtain that $Tr(F_{\tau \leq b}) \approx Tr(F^2)$; hence both of them are quasi-isomorphic to $Tr(F^1)$.

In particular, let $A = Ab$; let F be (contravariant) representable by some Y in some differential graded $K \supset J$ such that TrK possesses a weight structure extending w and its adjacent t -structure t . Then our reasoning shows that the objects $t_{\leq i}Y$ represent the truncated realizations for $Tr(K(-, Y))$ (in $Tr(K) \supset \mathfrak{H}$; up to quasi-isomorphism i.e. they give the cohomology groups required). This is a differential graded version of part 6 of Theorem

4.4.2. Besides, in this case the fact that the filtrations induced by the morphisms $X \rightarrow w_{\leq i}X$ and by $t_{\leq i}Y \rightarrow Y$ coincide also follows from part 6 of Theorem 4.4.2.

So, the results of §7.1 below yield that the truncated realizations both for the 'classical' (Weil) realizations of motives and for motivic cohomology are representable by objects of DM_{-}^{eff} . This fact seems to be far from obvious.

2. As was mentioned in part 4 of Remark 4.4.3, one could deduce the comparison of spectral sequences statement in the representable case from the remark above and Theorem 4.4.2. Moreover, one could construct a *nice duality* of \underline{C}, w with the t -structure on the category $\underline{D} = Tr(DG - Fun(J, B(A)))$ (differential graded functors) that corresponds to the canonical truncation of A -complexes; see Definition 2.5.1 of [Bon10] and part 5 of Remark 4.4.3 below. Also, \underline{D} could be called 'a category of functors $\underline{C} \rightarrow A$ '; see Remark 2.5.2. The realizations of the type considered above correspond to some objects of this category; truncations of a realization with respect to this t -structure would be exactly its truncated realizations (cf. Proposition 2.5.4 of [Bon10]).

6.5 $SmCor$, DM_{gm}^{eff} and DM_{gm} ; the Chow weight structure

We recall some definitions of [Voe00a].

k will denote our perfect ground field; we will mostly assume that the characteristic of k is zero. pt is a point, \mathbb{A}^n is the n -dimensional affine space (over k), x_1, \dots, x_n are the coordinates, \mathbb{P}^1 is the projective line.

$Var \supset SmVar \supset SmPrVar$ will denote the class of all varieties over k , resp. of smooth varieties, resp. of smooth projective varieties.

We define the category of smooth correspondences: $Obj SmCor = SmVar$, $SmCor(X, Y) = \bigoplus_U \mathbb{Z}$ for all $U \subset X \times Y$ that are integral closed finite subschemes which are dominant (over a connected component of) X . The elements of $SmCor(X, Y)$ are called *finite correspondences* from X to Y .

Remark 6.5.1. The composition of $U_1 \subset X \times Y$ with $U_2 \subset Y \times Z$ as in the definition of finite correspondences is defined as always in the categories of motives i.e. one considers the obvious scheme-theoretic analogue of $\{(x \in X, z \in Z) : \exists y \in Y : (x, y) \in U_1, (y, z) \in U_2\}$. Note that the composition is well-defined without any factorization by equivalence relations needed. Next one extends composition to all $SmCor(-, -)$ by linearity.

Note that this definition is compatible with the naive notion of composition of multivalued functions. Now, to $\sum c_i U_i \in SmCor(X, Y)$, $c_i \neq 0$ one could associate a multi-valued function whose graph is $\bigcup U_i$. Applying

this definition, one can define images and preimages of finite correspondences (and their restrictions). Below we will assume that images are closed integral subschemes of the corresponding Y 's.

$SmCor$ is additive: the addition of objects is given by the disjoint union operation for varieties. It is also a tensor category; the tensor product operation is given by the Cartesian product of varieties.

$Shv(SmCor)$ is the abelian category of those additive cofunctors $SmCor \rightarrow Ab$ that are sheaves in the Nisnevich topology.

$DM_-^{eff} \subset D^-(Shv(SmCor))$ is defined as the subcategory defined by the condition that the cohomology sheaves are homotopy invariant (i.e. $S(X) \cong S(X \times \mathbb{A}^1)$ for any $S \in SmVar$).

There is a natural functor $RC \circ L : K^b(SmCor) \rightarrow DM_-^{eff}$ (cf. Theorem 3.2.6 of [Voe00a]) given by Suslin complexes (see below); it can be factorized as the composition of the 'localization by homotopy invariance and Mayer-Vietoris' and a full embedding; its categorical image will be denoted by DM^s . One can restrict $RC \circ L$ to obtain a functor $M_{gm} : SmVar \rightarrow DM^s$. Moreover, one can extend M_{gm} to Var (see §4.1 of [Voe00a]); unfortunately, in the case $\text{char } k > 0$ one would have to take DM_-^{eff} as the target of this (extended) M_{gm} . Therefore, cohomology of varieties can be expressed in terms of cohomology of motives.

DM_-^{eff} is idempotent complete; hence it contains the idempotent completion of DM^s which is Voevodsky's DM_{gm}^{eff} (by definition; see [Voe00a]).

Now we define a differential graded category J with $Obj J = SmPrVar$ (the addition of objects is the same as for $SmCor$). The morphisms of J are given by *cubical Suslin complexes* $J_i(Y, P) \subset SmCor(\mathbb{A}^{-i} \times Y, P)$ consisting of correspondences that 'are zero if one of the coordinates is zero'. Being precise: consider $C^i(P, Y) = SmCor(\mathbb{A}^{-i} \times Y, P)$ for all $P, Y \in SmVar$; note that C^i are zero for positive i . For all $1 \leq j \leq -i$, $x \in k$, we define $d_{ijx} = d_{jx} : C^i \rightarrow C^{i+1}$ as $d_{jx}(f) = f \circ g_{jx}$, where $g_{jx} : \mathbb{A}^{-i-1} \times Y \rightarrow \mathbb{A}^{-i} \times Y$ is induced by the map $(x_1, \dots, x_{-1-i}) \rightarrow (x_1, \dots, x_{j-1}, x, x_j, \dots, x_{-1-i})$. We define $J^i(Y, P)$ as $\cap_{1 \leq j \leq -i} \text{Ker } d_{j0}$. The boundary maps $\delta^i : J^i(-, -) \rightarrow J^{i+1}(-, -)$ are defined as $\sum_{1 \leq j \leq -i} (-1)^j d_{j1}$.

The composition of morphisms in J is induced by the obvious composition $C^i(Y \times \mathbb{A}^{-j}, X \times \mathbb{A}^{-j}) \times C^j(Z, Y) \rightarrow C^{i+j}(Z, X)$ combined with the embedding of $C^i(Y, X)$ into $C^i(Y \times \mathbb{A}^{-j}, X \times \mathbb{A}^{-j})$ via 'tensoring' its elements by $id_{\mathbb{A}^{-j}}$; here $X, Y, Z \in SmPrVar$, $i, j \leq 0$.

It was checked in §2 of [Bon09] that J is a differential graded category. It is negative by definition.

We denote $Tr(J)$ by \mathfrak{H} . \mathfrak{H} is equivalent to DM^s (if $\text{char } k = 0$) by Theorem 3.1.1 of [Bon09].

By Proposition 6.2.1 we obtain that there exists a weight structure w in \mathfrak{H} ; hence it also gives a weight structure for DM^s . We have $\underline{Hw} = J'_0$ where J'_0 is the small envelope of $J_0 = HJ$ (cf. Definition 4.3.1 and part II2 of Theorem 4.3.2).

Remark 6.5.2. 1. In §4.1 of [Voe00a] the *motif with compact support* for any $X \in Var$ was defined as the Suslin complex of a certain sheaf $L^c(X)$. For a proper X we have $M_{gm}^c(X) = M_{gm}(X)$. However, in order to increase the chances to obtain a geometric motif (with compact support) one could define $M_{gm}^c(X)$ using Poincare duality; see Appendix B of [HuK06]. In the case $\text{char } k = 0$ these definitions coincide and yield an object of DM^s for any $X \in Var$.

In Theorem 6.2.1 of [Bon09] it was proved that for a smooth X we have $M_{gm}(X) \in DM^{sw \geq 0}$, $M_{gm}^c(X) \in DM^{sw \leq 0}$. Using the blow-up distinguished triangle (see Proposition 4.1.3 of [Voe00a]) one could also show that for a proper X we have $M_{gm}(X) = M_{gm}^c(X) \in DM^{sw \leq 0}$.

2. As in part 3 of Remark 2.1.3 one could consider semi-motives $W_i(M_{gm}(X))$ and $W_i(M_{gm}^c(X))$ for all $i \in \mathbb{Z}$, $X \in Var$; they lie in DM_{gm}^{eff} . We obtain that $W_0(M_{gm}(X)) = M_{gm}(X)_*$ for proper X , whereas $W_{-1}(M_{gm}(X)) = 0$ for $X \in SmVar$. Recall that (12) allows to express the weight filtration on the cohomology of (the motif of) X in terms of $W_i(M_{gm}(X))$.

In DM_{gm}^{eff} we have a decomposition $[P^1] = [pt] \oplus \mathbb{Z}(1)[2]$ for $\mathbb{Z}(1)$ being the *Tate motif*. Moreover, DM_{gm}^{eff} is a tensor category with $\otimes \mathbb{Z}(1)$ being a full embedding of DM_{gm}^{eff} into itself (the *Cancellation Theorem*, see Theorem 4.3.1 of [Voe00a] and [Voe10]). Hence one could define Voevodsky's DM_{gm} as the direct limit of DM_{gm}^{eff} with respect to tensoring by $\mathbb{Z}(1)$; it also could be described as the union of $DM_{gm}^{eff}(-i)$ (whereas each $DM_{gm}^{eff}(-i)$ is isomorphic to DM_{gm}^{eff}).

Proposition 6.5.3. *w extends to a weight structure for DM_{gm}^{eff} and DM_{gm} .*

Proof. I Extending w to DM_{gm}^{eff} .

We define $DM_{gm}^{eff w \leq 0}$ as the set of retracts of $DM^{sw \leq 0}$ in DM_{gm}^{eff} ; the same for $DM_{gm}^{eff w \geq 0}$. By Proposition 5.2.2, this gives a weight structure on DM_{gm}^{eff} .

II Extending w to DM_{gm} .

We note that tensoring by $\mathbb{Z}(1)[2]$ sends $[P]$ to a retract of $[P \times \mathbb{P}^1]$. Hence $\otimes \mathbb{Z}(1)[2]$ maps $DM_{gm}^{eff w \leq 0}$ and $DM_{gm}^{eff w \geq 0}$ into themselves. It follows that one can define $DM_{gm}^{w \leq 0}$ and $DM_{gm}^{w \geq 0}$ as $\cup DM_{gm}^{eff w \leq 0}(-i)[-2i]$ and $\cup DM_{gm}^{eff w \geq 0}(-i)[-2i]$, respectively. Indeed, the Cancellation Theorem gives us orthogonality; since each object of DM_{gm} belongs to $DM_{gm}^{eff}(-i) = DM_{gm}^{eff}(-i)[2i]$ for some $i \in \mathbb{Z}$, we also have the weight decomposition property.

□

Remark 6.5.4. Note that (for any \underline{C}) if w is bounded then $\underline{C}^{w \leq 0}$ consists exactly of objects that can be 'decomposed' into a weight Postnikov tower (see Definition 1.5.8) with $X^k = 0$ for $k > 0$; for $X \in \underline{C}^{w \geq 0}$ we can assume that $X^k = 0$ for $k < 0$.

Besides (see Proposition 6.2.1) for $\underline{C} = DM^s$ we can assume that all X^k can be presented as $M_{gm}(P^k)$ for $P^k \in SmPrVar$. For $\underline{C} = DM_{gm}^{eff}$ or $\underline{C} = DM_{gm}$ we have $P^k \in ObjChow \subset ObjDM_{gm}$ (see §6.6 below).

We call the weight structure constructed the *Chow* weight structure (for any of \mathfrak{H} , DM^s , DM_{gm}^{eff} , DM_{gm} , and also for DM_-^{eff} considered below).

6.6 The heart of the Chow weight structure

Now we calculate the hearts of w in each of the categories constructed.

First we recall the definition of (homological) Chow motives. In [Voe00a] it was proved that *Chow* can be described in the following way. One considers $Corr_{rat} = J_0$; this is (essentially) the usual category of rational correspondences. The category $Chow^{eff}$ is the idempotent completion of $Corr_{rat}$; it was shown in Proposition 2.1.4 of [Voe00a] that $Chow^{eff}$ is naturally isomorphic to the usual category of effective homological Chow motives. Moreover, the natural functor $Chow^{eff} \rightarrow DM_{gm}^{eff}$ is a full embedding (of additive categories). Note that $Chow^{eff}$ is a tensor category.

$Chow$ will denote the whole category of Chow motives i.e. $Chow[\mathbb{Z}(-1)[-2]]$. Note that the Chow motif that we denote by $\mathbb{Z}(1)[2]$ (this is compatible with the embedding $Chow \rightarrow DM_{gm}^{eff}$) was denoted by $\mathbb{Z}(1)$ by some authors.

So, the heart of the Chow weight structure for DM^s is the small envelope of $Corr_{rat}$ (note that it contains $\mathbb{Z}(1)[2]$ whereas $Corr_{rat}$ does not). Now Proposition 5.2.2 implies that the heart of DM_{gm}^{eff} is the idempotent completion of $Corr_{rat}$ i.e. the whole category $Chow^{eff}$. Lastly, we obtain that the heart of DM_{gm} equals $Chow$.

We obtain that for **any** (co)homological functor from DM_{gm}^{eff} (or DM_{gm}) there exist (Chow-)weight spectral sequences and weight filtrations. Note that we don't need any enhancements here (in contrast to [Bon09])! Moreover, Chow-weight spectral sequences are functorial with respect to all natural transformations of (co)homological functors (so, we also do not need any transformations for enhancements).

Remark 6.6.1. The same arguments as above also prove the existence of weight structures on rational hulls of DM^s , DM_{gm}^{eff} and DM_{gm} (i.e. we tensor the groups of morphisms by \mathbb{Q}) as well as on their idempotent completions (which do not coincide with $DM_{gm}^{eff} \otimes \mathbb{Q}$ and $DM_{gm} \otimes \mathbb{Q}$). If we denote

the latter by $DM_{gm}^{eff}\mathbb{Q} \subset DM_{gm}\mathbb{Q}$, then their hearts will be $Chow^{eff}\mathbb{Q} \subset Chow\mathbb{Q}$ (i.e. the idempotent completions of rational hulls). Note that in these statements one could take a perfect field k (of any characteristic, since the use of the resolution of singularities in the proofs can be replaced by de Jong's alterations, see [dJo96]). See §8.3 of [Bon09] and §7.3 below for more details.

Moreover, recent (unpublished) results of O. Gabber imply that for $\text{char } k = p$ one can prove our results for motives with $\mathbb{Z}[1/p]$ -coefficients. The author will treat this matter in a forthcoming paper.

Alternatively, one could also consider motives with $\mathbb{Z}/n\mathbb{Z}$ -coefficients (for some $n > 1$ and prime to p).

Lastly we note (as we also did in [Bon09]) that the results obtained (the existence of weight filtrations and of Chow-weight spectral sequences) also concern motivic cohomology of motives; cf. part 2 of Remark 2.4.3.

7 New results on motives

The first subsection is dedicated to the study of DM_-^{eff} . We prove that the Chow weight structure extends to it; DM_-^{eff} also supports a *Chow t -structure* t_{Chow} that is (left) adjacent to it. It follows that the Chow t -truncations of those objects that represent the 'classical' realizations of motives (or motivic cohomology) represent their *truncated realizations*; see Remark 6.4.1.

In §7.2 we note that any construction of the mixed motivic t -structure on $DM_{gm}^{eff}\mathbb{Q}$ would automatically yield a canonical weight filtration for the objects of its heart (i.e. for mixed motives).

In §7.3 we prove that a certain (possibly, 'infinite') weight complex functor can be defined for motives over any perfect field (without any resolution of singularities assumptions).

In §7.4 we apply the philosophy of adjacent structures to express the cohomology of a certain motif X with coefficients in the homotopy (t -structure) truncations of any $H \in Obj DM_-^{eff}$ in terms of the limit of H -cohomology of certain 'submotives' of X . Luckily, to this end (instead of the *Gersten weight structure* that is constructed in [Bon10] only in the case of a countable k) it suffices to have Gersten resolutions for homotopy invariant pretheories (constructed in [Voe00b]). These calculations are closely related with the well-known calculations of the E_2 -terms of the coniveau spectral sequence (by Bloch and Ogus in [BOg94]; see also [C-THK] and [Par96]).

In §7.5 we recall that (by the Beilinson-Lichtenbaum conjecture which was recently proved) torsion motivic cohomology is the (homotopy) truncation of the (torsion) étale one. Hence one can express torsion motivic cohomology

of certain motives in terms of étale cohomology of their 'submotives'. In particular, we obtain a formula for the (torsion) motivic cohomology with compact support of a smooth quasi-projective variety.

In §7.6 we calculate t_{Chow} in certain 'simple' cases; it turns out that it is closely related with unramified cohomology!

7.1 Chow weight and t -structures for DM_-^{eff}

We recall (see §3 of [Voe00a]) that for any $S \in DM_-^{eff}$ and $X \in SmVar$ we have $DM_-^{eff}(M_{gm}(X), S) = \mathbb{H}^0(S)(X)$ (here S is considered as a complex of sheaves). It follows (cf. §6.5) that $M_{gm}(X)$ for $X \in SmPrVar$ weakly generate DM_-^{eff} .

Now we take $\{C_i\} = ObjChow \subset ObjDM_-^{eff}$ (we can assume that $ObjChow$ is a set). We obtain that $(DM_-^{eff}, \{C_i\})$ satisfy the conditions of part I1 of Theorem 4.5.2. Hence it has a t -structure whose heart is $Chow_*^{eff}$; we will denote it by t_{Chow} . Unfortunately, it seems that t_{Chow} cannot be restricted to DM_{gm}^{eff} (i.e. it is not 'geometric').

Remark 7.1.1. Recall that for any $X \in ObjDM_{gm}^{eff} \subset ObjDM_-^{eff}$ there is a natural exact functor $\underline{Hom}(X, -) : DM_-^{eff} \rightarrow DM_-^{eff}$ defined (see Remark 14.12 of [MVW06]). Since tensor products (in DM_-^{eff}) of Chow motives are Chow motives also, we obtain (cf. Proposition 4.4.5): for any $X \in ObjChow \subset ObjDM_{gm}^{eff}$ we can restrict $\underline{Hom}(X, -)$ to an exact functor $Ht_{Chow} \rightarrow Ht_{Chow}$. In particular, Ht_{Chow} admits certain 'negative Tate twists' (defined by $\underline{Hom}(\mathbb{Z}(1)[2], -)$).

Note (in contrast) that the homotopy t -structure for DM_-^{eff} (see below) is respected by $\underline{Hom}(\mathbb{Z}(1)[1], -)$.

Now we check that the Chow weight structure of DM_{gm}^{eff} can be extended to DM_-^{eff} . Till the end of this section $\underline{C} = DM_-^{eff}$, t will denote the homotopy t -structure of DM_-^{eff} (defined as in [Voe00a]). This is the t -structure corresponding to Nisnevich hypercohomology i.e. $X \in \underline{C}^{t \leq 0}$ (resp. $X \in \underline{C}^{t \geq 0}$) whenever its Nisnevich hypercohomology is concentrated in non-positive (resp. non-negative) degrees. Note that in all $\underline{C}^{t \leq i}$ arbitrary coproducts exist.

We define $\underline{C}^{w \leq 0}$ as the Karoubi-closure in \underline{C} of the closure of $DM_{gm}^{effw \leq 0}$ in \underline{C} with respect to arbitrary coproducts and 'extensions' (as in Definition 1.3.1). Note that $\underline{C}^{w \leq 0} \subset \underline{C}^{t \leq 0}$. We recover $\underline{C}^{w \geq 0}$ from $\underline{C}^{w \leq 0}$ via the orthogonality condition (in the usual way, see part 4 of Lemma 1.3.7). $\underline{C}^{w \geq 0}$ is extension-stable (see part 1 of Lemma 1.3.7). Besides, it contains arbitrary coproducts of objects of $DM_{gm}^{effw=0}$ (here we apply the compactness of

objects of DM_{gm}^{eff} in \underline{C}).

As usual, the only non-trivial axiom check here is the verification of the existence of weight decompositions. Recall that any object of $Shv(SmCor)$ has a certain canonical resolution by direct sums of $L(X) = SmCor(-, Y)$ for $Y \in SmVar$ (placed in degrees ≤ 0 ; see §3.2 of [Voe00a]). Hence any object X of DM_{-}^{eff} is a homotopy colimit of certain X_i ($i \in \mathbb{Z}$) such that: the cone of $X_i \rightarrow X_{i+1}$ is a coproduct of some $M_{gm}(Y_{ij})[i]$; $X_l = 0$ for some $l \in \mathbb{Z}$. The limit of X_i equals X indeed by Lemma 4.2.6.

We construct $Z = X^{w \geq 1}$ as a homotopy colimit of $X_l^{w \geq 1}$ (see Definition 4.2.1). Note that a weight decomposition of X_l could be constructed using any possible weight decompositions of $\coprod M_{gm}(Y_{ij})[i]$ (see Remark 1.5.5).

We should check that the colimit exists in DM_{-}^{eff} . For any $Y \in SmVar$, $i > 0$, we have $(M_{gm}(Y)[i])^{w \geq 1} \in \underline{C}^{t \leq 0}$ (for any choice of $(M_{gm}(Y)[i])^{w \geq 1}$). This is easy since $M_{gm}(Y)[i] \in \underline{C}^{t \leq -i}$ and $(M_{gm}(Y)[i])^{w \leq 0} \in \underline{C}^{w \leq 0} \subset \underline{C}^{t \leq 0}$. Combining these statements for all Y_{ij} and i yields the boundedness required.

We have the composition morphisms $X_l^{w \geq 1} \rightarrow X_l[1] \rightarrow X[1]$; by part 1 of Lemma 4.2.3 this system of morphisms can be lifted to some morphism $Z \rightarrow X[1]$. We should check that it yields a weight decomposition (if we make the choices in the construction in a 'clever' way). By Lemma 4.2.4 we can assume that $Z \in \underline{C}^{w \geq 0}$. We denote a cone of $Z[-1] \rightarrow X$ by Y .

Now, it suffices (see part 1 of Remark 1.3.4) to check that the induced map $\underline{C}(C, Z) \rightarrow \underline{C}(Z, X[1])$ is an isomorphism for any $C \in \underline{C}^{w \geq 1}$ and is surjective for $C \in \underline{C}^{w \geq 0}$.

First suppose that for some $i \in \mathbb{Z}$ we have $C \perp R$ for any $R \in \underline{C}^{t \leq i}$. Then the sequence $\underline{C}(C[1], X_i)$ stabilizes; this yields the result required by part 2 of Lemma 4.2.3. Hence for any such C we have $C \perp Y[1]$.

We denote ${}^\perp(Y[1]) \cap \underline{C}^{w \geq 0}$ by S . We should check that $S = \underline{C}^{w \geq 0}$. Certainly, S is extension-stable and closed with respect to arbitrary coproducts (in \underline{C}).

We have $DM_{gm}^{eff w \geq 0} \subset S$. Indeed, any $C \in Obj DM_{gm}^{eff}$ is a retract of an object that can be obtained from (a finite number of) motives of smooth varieties by considering cones of morphisms; whereas for $X \in SmVar$ we have $DM_{-}^{eff}(M_{gm}(X), R) = \{0\}$ for any $R \in \underline{C}^{t \leq -\dim X - 1}$ (since the Nisnevich cohomological dimension of a scheme is not greater than its dimension). Next, all coproducts of objects of $DM_{gm}^{eff w \geq 0}$ (belonging to DM_{-}^{eff}) also belongs to S . Therefore, it suffices to prove that any object of $\underline{C}^{w \geq 0}$ can be 'approximated' by such coproducts.

By the same method as above, we present $C \in \underline{C}^{w \geq 0}$ as a homotopy colimit of certain C_i for the cone of $C_i \rightarrow C_{i+1}$ being a coproduct of some $M_{gm}(E_{ij})[i]$; $C_l = 0$ for some $l \in \mathbb{Z}$.

Since any coproduct of distinguished triangles is a distinguished triangle, we can construct distinguished triangles $(\coprod M_{gm}(E_{ij})[i])^{\geq 0} \rightarrow A_i \rightarrow B_i$ for $A_i \in \underline{C}^{w \leq 0}$ and $B_i \in S$ (they will be coproducts of objects of DM_{gm}^{eff}). Next, applying Remark 1.5.5 for $D = \underline{C}^{w \leq 0}$, $E = S$, we can (starting from C_l) inductively construct distinguished triangles $C_i \rightarrow F_i \rightarrow G_i$ for $F_i \in \underline{C}^{w \leq 0}$, $G_i \in S$. We also construct distinguished triangles $C_i[-1] \rightarrow L_i \rightarrow M_i$ for $L_i \in \underline{C}^{w \leq 0}$, $M_i \in S$.

By Definition 4.2.1, we have a distinguished triangle $\coprod C_i \rightarrow \coprod C_i \rightarrow C$. Now note that $\coprod F_i$, $\coprod G_i$, $\coprod L_i$ and $\coprod M_i$ exist in \underline{C} (since by the same argument as the one used above all of the summands belong to $\underline{C}^{t \leq l}$ for some $l \in \mathbb{Z}$). Since any coproduct of distinguished triangles in \underline{C} is a distinguished triangle and $\underline{C}^{w \leq 0}$ and S are closed with respect to all coproducts, we obtain distinguished triangles $\coprod C_i \rightarrow \coprod F_i \rightarrow \coprod G_i$ and $\coprod C_i[-1] \rightarrow \coprod L_i \rightarrow \coprod M_i$ with $\coprod F_i, \coprod L_i \in \underline{C}^{w \leq 0}$, $\coprod G_i, \coprod M_i \in S$.

Applying Remark 1.5.5 again we obtain a distinguished triangle $C[-1] \xrightarrow{f} U \rightarrow V$ for some $U \in \underline{C}^{w \leq 0}$ and $V \in S$. Hence $f = 0$; therefore C is a retract of V . Thus $C \in S$.

Remark 7.1.2. Unfortunately, one cannot define a weight structure for DM_{-}^{eff} that would be left adjacent to the homotopy t -structure. Indeed, for an object X of this heart the functor $DM_{-}^{eff}(X, -)$ should be exact on the category of homotopy invariant sheaves with transfers. So the heart should contain 'motives of points' i.e. motives of local smooth k -algebras; the latter are (usually) pro- k -varieties and not varieties. Yet in §4.1 of [Bon10] we define the corresponding *Gersten* weight structure in a certain triangulated $\mathfrak{D}_s \supset DM_{gm}^{eff}$; see also part 4 of Remark 7.4.3 below.

7.2 Weight filtration for (conjectural) mixed motives

Suppose now that there exists a so-called mixed motivic t -structure t_{MM} on DM_{gm}^{eff} or $DM_{gm}^{eff}\mathbb{Q}$ (then one could extend it to DM_{gm} and $DM_{gm}\mathbb{Q}$, respectively). We will not discuss its properties here (until §8.6); however it would automatically induce a homological functor $H_{MM} : DM_{gm}^{eff} \rightarrow MM$ for some abelian category MM (of so-called mixed motives) that is the heart of the t -structure. Hence for any $X \in DM_{gm}^{eff}$ there will be a certain (weight) filtration on $H_{MM}^i(X)$. This filtration would be trivial (i.e. 'canonical') when X is smooth projective. It can be easily checked that there could exist only one filtration on $H_{MM}^i(X)$ which is DM_{gm}^{eff} -functorial and satisfies this property.

Moreover, any transformation $H_{MM} \rightarrow H$ for H being a realization (of DM_{gm}^{eff}) with values in an abelian category would induce the transformation

of the weight filtration for H_{MM} to the weight filtration of H . Here the weight filtration of H is defined by the weight structure method, yet it coincides with the 'classical' one (cf. part 2 of Remark 2.4.3).

Therefore we obtain that our results will give a certain weight filtration for $H_{MM}^i(X)$ (and the corresponding Chow-weight spectral sequence) automatically when H_{MM} will be defined. Note we don't need any information on H_{MM} for this! However this construction does not yield automatically that the filtration on $H_{MM}^i(X)$ obtained depends only on the object $H_{MM}^i(X)$ and does not depend on the choice of X .

Alternatively, one can obtain weights for $X \in MM$ by presenting it as $H_{MM}^0(X)$ (so we use the embedding $MM = \underline{Ht}_{MM} \rightarrow DM_{gm}\mathbb{Q}$). Then one obtains a weight filtration for X that does not depend on any choices. This filtration certainly should coincide with those given by the previous method; yet in order to prove this one needs to know that t_{MM} -truncations of Chow motives are Chow motives themselves. This is certainly expected to be true (it corresponds to so-called Chow-Kunneth decompositions of motives of varieties). We will say more on weights for mixed motives in §8.6 below.

7.3 Motives over perfect fields of finite characteristic

In our study of motives (here and in [Bon09]) we applied several results of [Voe00a] that use resolution of singularities. So we had to assume that the characteristic of the ground field k is 0. In §8.3 of [Bon09] it was shown that using de Jong's alterations one can extend most of our results to motives with rational coefficients over an arbitrary perfect k .

In this subsection (and also in all remaining parts of this section) we consider motives with integral coefficients over a perfect field k of characteristic 0. Our goal is to justify a certain claim made in §8.3.1 of [Bon09].

In [BeV08] it was proved unconditionally that DM^s has a differential graded enhancement. In fact, this fact can be easily obtained by applying Drinfeld's description of localizations of enhanced triangulated categories. Moreover, Proposition 5.6 of [BeV08] extends the Poincaré duality for Voevodsky motives to our case. Therefore for $P, Q \in SmPrVar$ we obtain

$$DM^s(M_{gm}(P), M_{gm}(Q)[i]) = Corr_{rat}([P], [Q]) \text{ for } i = 0; \text{ } 0 \text{ for } i > 0.$$

Hence the triangulated subcategory DM_{pr} of DM^s generated by $[P]$, $P \in SmPrVar$ could be described as $Tr(I)$ for a certain negative differential graded I . In particular, we obtain the existence of a conservative weight complex functor $t_0 : DM_{pr} \rightarrow K^b(Corr_{rat})$. Moreover, for any realization of DM_{pr} and any $X \in Obj DM_{pr}$ one has the Chow-weight spectral sequence T .

The problem is that (to the knowledge of the author) at this moment there is no way known to prove that DM_{pr} contains the motives of all smooth varieties (though it contains the motives of those varieties that admit 'nice compactifications').

Instead we will prove that the weight structure on DM_{pr} can be extended to a weight structure on a larger category containing all $M_{gm}(X)$.

Recall that M_{gm} is a full embedding of $DM_{gm}^{eff} \supset DM_{pr}$ into DM_-^{eff} , whereas $DM_-^{eff} \subset D(Shv(SmCor))$ (M_{gm} is denoted by i in Theorem 2.3.6 of [Voe00a]). We denote by $D \subset D(Shv(SmCor))$ the full category of complexes with homotopy invariant cohomology sheaves. We have a full embedding $DM_-^{eff} \rightarrow D$.

We can extend to $D \subset D(Shv(SmCor))$ the assertion of Proposition 3.2.3 of [Voe00a] (i.e. construct a projection $D(Shv(SmCor)) \rightarrow D$ which is left adjoint to the embedding) using the fact that $D(Shv(SmCor))^{t \leq 0} \perp D(Shv(SmCor))^{t \geq 1}$. Here t denotes the usual t -structure of $D(Shv(SmCor))$ (corresponding to the homotopy t -structure for DM_-^{eff}). It follows that all objects of $M_{gm}(DM_{gm}^{eff})$ are compact. Indeed, it is sufficient to prove this for $M_{gm}([X])$ where $X \in SmVar$; Proposition 3.2.3 of [Voe00a] implies that $D(M_{gm}(X), -)$ is the corresponding hypercohomology functor which commutes with arbitrary coproducts.

Consider $S = \{X \in Obj D : Y \perp X \ \forall Y \in Obj DM_{pr}\}$. Note that in the definition of S it suffices to consider $Y = M_{gm}(P)[i]$, $P \in SmPrVar$, $i \in \mathbb{Z}$, since $[P]$ generate DM_{pr} . Obviously, S is the class of objects for a certain full triangulated subcategory of $D(Shv(SmCor))$. We denote the localization (i.e. the Verdier quotient) of D by S by D_S . By definition of S , the set $H = \{[P], P \in SmPrVar\}$ weakly generates D_S . Since objects of DM_{pr} are compact, S is closed with respect to arbitrary coproducts. It follows that D_S admits arbitrary coproducts. Note that $DM_{pr} \subset D_S$ by Proposition III.2.10 of [GeM03]; hence we have a full embedding $Chow^{eff} \rightarrow D$.

By part I2 of Theorem 4.5.2 we obtain that D_S supports adjacent weight and t -structures which we will call Chow ones. By part II of Theorem 4.5.2 we have $\underline{H}t_{Chow} = Chow_*^{eff}$. Moreover, $\underline{H}w$ is the category $Chow_{\oplus}^{eff}$ of arbitrary coproducts of effective Chow motives since $Chow^{eff}$ is idempotent complete.

Note that the definition of w_{Chow} is compatible with the definition of the Chow weight structure on DM_{pr} . In particular, this reasoning extends the weight complex functor to a functor $D \rightarrow K_{\mathfrak{w}}(Chow_{\oplus}^{eff})$. This would give a (possibly, infinite) weight complex for any $X \in Obj DM_{gm}^{eff}$. Recall that (by the results of §8.3.2 of [Bon09]) $t(X)$ becomes (homotopy equivalent to) a finite complex after tensoring the coefficients by \mathbb{Q} . This weight

complex functor can be 'strengthened' (see part 2 of Remark 6.2.2) since $D(Shv(SmCor))$ has a differential graded enhancement.

7.4 Coniveau and truncated cohomology

Let k be an arbitrary perfect field, $H \in Obj DM_-^{eff}$. We denote $\tau_{\leq i} H$ by H' (i is fixed, τ is the t -truncation with respect to the homotopy t -structure). We denote by H'' the 'complement of H' to H ' i.e. $H'' = Cone(H' \rightarrow H)$. Note that the cohomology of H'' is concentrated in degrees $> i$. $j \in \mathbb{Z}$ will be a fixed integral number up to the end of the section.

Let

$$M = U_u \xrightarrow{d_u} U_{u-1} \xrightarrow{d_{u-1}} \dots U_1 \xrightarrow{d_1} U_0 \in Obj DM_{gm}^{eff} \quad (32)$$

be a complex in $SmCor$ (U_l is in degree $-l$). We demand that for all r , any (closed) point $u \in U_{r-1}$ the codimension of the preimage (in the sense of Remark 6.5.1) $\text{codim}_{U_r} d_r^{-1}(u) \geq \text{codim}_{U_{r-1}} u - 1$; here we define the codimension of a subvariety as the minimum of codimensions of its parts in the corresponding connected components.

We fix some $j \in \mathbb{Z}$, DM_-^{eff} will be denoted by \underline{C} . In order to write a formula for the H' and H'' -cohomology of M and prove it we will need some notation and certain orthogonality statements.

Let (Y_l^0, Y_l^1) run through open subschemes of U_l such that: $U_l \setminus Y_k$ is everywhere of codimension $\geq j - i - k + 1 - l$ in U_l ($k = 0, 1, 0 \leq l \leq u$), the images (in the sense of Remark 6.5.1) $d_l(Y_l^k) \subset d_l(Y_{l-1}^k)$ for all k, l . We define the motives $L^k = Y_u^k \rightarrow \dots \rightarrow Y_1^k \rightarrow Y_0^k$ for $k = 0, 1$ using the corresponding restrictions of d_l . Note that if $Y_l^1 \subset Y_l^0$ for all l then we have natural morphisms $L^1 \rightarrow L^0 \rightarrow M$. We denote $N^k = Cone(L^k \rightarrow M) \in Obj DM_{gm}^{eff}$.

Lemma 7.4.1. 1. $\underline{C}(N^k, H''[r]) = 0$ for any $r \leq j - k + 1$ and any Y_l^k .

2. $\varinjlim \underline{C}(L^1, H'[j+1]) = \varinjlim \underline{C}(L^1, H'[j]) = \varinjlim \underline{C}(L^0, H'[j+1]) = \{0\}$.

Proof. 1. It easily seen (by cutting $H[j]$ into its t -cohomology) that it suffices to prove a similar statement for H'' replaced by any homotopy invariant $S \in Shv(SmCor)$ shifted by $v \leq j - k$.

First let all U_l except U_t be empty (and all $d_l = 0$). Then our (last) assertion could be easily deduced from Lemma 4.36 of [Voe00b] (and some cohomological comparison results of Voevodsky) using the standard coniveau spectral sequence argument. We write down a (short) proof here. The classical coniveau spectral sequence could be found, for example, in [C-THK]: for any $U \in SmVar$ by formula (1.2) of ibid. there exists a spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in U^{(p)}} H_{x, Zar}^{p+q}(U, S) \implies H_{Zar}^{p+q}(U, S);$$

here $U^{(p)}$ denotes the set of points of U of codimension p , $H_{x,Zar}^{p+q}(U, S)$ is the local Zariski cohomology group (see §4.6 of [Voe00b] and Lemma 1.2.1 of [C-THK]). Note that this spectral sequence is functorial with respect to open embeddings. Now, Lemma 4.36 of [Voe00b] yields cohomological purity in this case; in particular, $H_{x,Zar}^{p+q}(U, S) = 0$ for any $x \in U^{(p)}$ unless $q = 0$. It follows that the map $H_{Zar}^v(U_t, S) \rightarrow H_{Zar}^v(Y_t^k, S)$ is bijective for $v < j - k - t$ and injective for $v = j - k - t$. Next, the Zariski cohomology of S coincides with its Nisnevich cohomology by Theorem 5.3 of [Voe00b], whereas the latter equals $\underline{C}(-, S[v])$ by Proposition 3.2.3 of [Voe00a]. Hence the long exact sequence of relative cohomology for (Y_t^k, Y_t) yields our (last) claim in this case.

Now let $u = 1$. We have an exact sequence

$$\{0\} = \underline{C}(Y_1^k \rightarrow U_1, S[v-1]) \rightarrow \underline{C}(N^k, S[v]) \rightarrow \underline{C}(Y_0^k \rightarrow U_0, S[v]) = \{0\}$$

for $k = 0, 1$, $v \leq j - k$; this yields the claim in this case. The case $u > 1$ can be easily obtained from similar exact sequences by induction.

2. The proof is similar to those of part 1. One should cut H' into its t -pieces and apply the coniveau spectral sequence arguments.

To this end we recall that the inductive limit of (long) exact sequences is exact, so we can pass to the limit in the coniveau spectral sequence. Besides, the codimension condition (on U_l) implies that for sets of Y_l^k as in the assertion all (single) Y_l^k could be 'as small as possible'. This means that for any open $Y \subset U_l$ such that $U_l \setminus Y$ is everywhere of codimension $\geq j - i - k + 1 - l$ in U_l ($k = 0, 1$) can be completed to some set of Y_l^k ; besides, we can intersect such sets (componentwisely). It follows that the corresponding $\varinjlim_{Y_l^k} \bigoplus_{x \in Y_l^{k(p)}} H_{x,Zar}^p(U, S) = \{0\}$ since $\varprojlim_{Y_l^k} Y_l^{k(p)} = \emptyset$ (for the values of p corresponding to our situation).

□

Theorem 7.4.2. *I We have an isomorphism*

$$\underline{C}(M, H''[j]) \cong \text{Im}(\varinjlim \underline{C}(L^0, H[j]) \rightarrow \varinjlim \underline{C}(L^1, H[j])). \quad (33)$$

Here connections between the cohomology of L^k for different sets (Y_l^k) are induced by open embeddings of varieties.

II For any j we have:

$$\underline{C}(M, H'[j]) \cong \text{Im}(\varinjlim \underline{C}(N^0, H[j]) \rightarrow \varinjlim \underline{C}(N^1, H[j])), \quad (34)$$

the limit is defined as in assertion I.

III 1. The isomorphisms described above are functorial in the obvious way with respect to 'nice' morphisms of complexes of correspondences $(f_l) : M' \rightarrow$

M . Here M_l is a complex of U_l' , (f_l) is nice if for any l , for any (closed) point $u \in U_l$ we have $\text{codim}_{U_l'} f_l^{-1}(u) \geq \text{codim}_{U_l} u$ (in the sense of Remark 6.5.1).

2. Furthermore, suppose that for some (f_l) and fixed set of $Y_l^0 \subset U_l$ (satisfying the above conditions) we have $\text{codim}_{U_l'} f_l^{-1}(U_l \setminus Y_l^0) \geq j - i - l$. Then the morphism $f_{H''}^* : H''(M) \rightarrow H''(M')$ (resp. $f_{H'}^* : H'(M) \rightarrow H'(M')$) is compatible with the natural morphism $\underline{C}(L^0, H[j]) \rightarrow \underline{C}(L^{1'}, H[j])$ (resp. $\underline{C}(N^0, H[j]) \rightarrow \underline{C}(N^{1'}, H[j])$) via the isomorphism of assertion I (resp. assertion II).

Proof. I Shifting H we easily reduce the statement to the case $i = 0$.

Lemma 7.4.1 allows us to argue similarly to the proof of assertion 7 of Theorem 4.4.2; note that our assertion is an analogue of part 8 of loc.cit.

Part 1 of the lemma yields exact sequences

$$\begin{aligned} \{0\} &= \varinjlim \underline{C}(N^0, H''[j]) \rightarrow \underline{C}(M, H''[j]) \\ &\rightarrow \varinjlim \underline{C}(L^0, H''[j]) \rightarrow \varinjlim \underline{C}(N^0, H''[j+1]) = \{0\} \end{aligned} \quad (35)$$

and

$$\{0\} = \varinjlim \underline{C}(N^1, H''[j]) \rightarrow \underline{C}(M, H''[j]) \rightarrow \varinjlim \underline{C}(L^1, H''[j]) \rightarrow \dots \quad (36)$$

By part 2 of Lemma 7.4.1 we also have $\varinjlim \underline{C}(L^1, H[j]) \cong \varinjlim \underline{C}(L^1, H''[j])$. Now we consider the commutative diagram

$$\begin{array}{ccccc} \varinjlim \underline{C}(L^0, H[j]) & \longrightarrow & \varinjlim \underline{C}(L^1, H[j]) & & \\ \downarrow g & & \downarrow h & & \\ \underline{C}(M, H''[j]) & \xrightarrow{t} & \varinjlim \underline{C}(L^0, H''[j]) & \xrightarrow{p} & \varinjlim \underline{C}(L^1, H''[j]) \end{array} \quad (37)$$

We have proved that t and h are bijective, g is surjective, and p is injective. This immediately yields the assertion required.

II Using Lemma 7.4.1, we can argue exactly as in the proof of assertion 7 of Theorem 4.4.2 (and dually to the reasoning above).

III1. We describe the functoriality in question for 'nice' (f_l) .

For $r = 0$ or 1 let Y_l^r be fixed for all l . Then we can take $Y_l^{r'} = U_l' \setminus f_l^{-1}(U_l \setminus Y_l^r)$; (f_l) induces morphisms $L^{r'} \rightarrow L^r$ and $N^{r'} \rightarrow N^r$. It remains to note that the proofs of assertions I and II are compatible with these morphisms.

III2. We check the compatibility desired for H'' ; the statement for H' could be proved similarly (and dually in the categorical sense).

We should check the following. Let $v \in \underline{C}(M, H''[j])$ come from some $w \in \underline{C}(L^0, H[j])$ (for our fixed Y_l^0) via (33). Denote by (33') the isomorphism

(33) with M replaced by M' , L^r replaced by $L^{r'}$. Then $f_{H''}^*(v)$ should be mapped via (33') to the image of $f_H^*(w)$ in $\varinjlim \underline{C}(M^{1'}, H[j])$. Here f_H^* is the map $\underline{C}(L^0, H[j]) \rightarrow \underline{C}(L', H[j])$ induced by (f_l) , and L' is the complex of $U'_l \setminus f_l^{-1}(U_l \setminus Y_l^0)$.

The latter fact follows easily from the commutativity of the diagrams

$$\begin{array}{ccccc} \underline{C}(M, H''[j]) & \longrightarrow & \underline{C}(L^0, H''[j]) & \longrightarrow & \varinjlim \underline{C}(L^0, H''[j]) \\ \downarrow & & \downarrow & & \\ \underline{C}(L', H''[j]) & \longrightarrow & \underline{C}(L', H''[j]) & \longrightarrow & \varinjlim \underline{C}(L^1, H''[j]) \end{array}$$

and

$$\begin{array}{ccc} \underline{C}(L^0, H[j]) & \longrightarrow & \underline{C}(L^0, H''[j]) \\ \downarrow & & \downarrow \\ \underline{C}(L', H[j]) & \longrightarrow & \underline{C}(L', H''[j]) \end{array}$$

□

Remark 7.4.3. 1. The main difference of this result from the usual comparison of spectral sequences (as in [Par96]) is: we calculate the D -terms of the corresponding exact couple instead of E -ones; we compute cohomology of certain motives (instead of varieties as in [BOg94] and [C-THK]).

2. Instead of applying assertion III2 to a single 'nice' set of Y_l^0 one could consider a (directed) system of those. This is especially actual if the right hand sides of (33) or (34) can be calculated using such a 'nice' directed subset of the set of all possible (Y_l^0) (which is often the case). In this case part III2 allows to 'calculate' $f_{H''}^*$ (resp. $f_{H'}^*$) completely.

3. One could generalize (34) in the following way. Let $r \geq i$, denote $\tau_{\leq r} H$ by G . Then for the corresponding morphism of cohomology theories $H' \rightarrow G$ we have

$$\mathrm{Im}(G^j(M) \rightarrow H'[j](M)) \cong \varinjlim \mathrm{Im}(H^j(N^0) \rightarrow H^j(N^{r-i+1})). \quad (38)$$

Here N^{r-i+1} is defined similarly to N^0, N^1 in the theorem. This statement could be easily obtained by calculating the D -terms of the higher derived couples for the coniveau spectral sequence; cf. Theorem 4.4.2.

4. Instead of considering limits of cohomology of motives we could have considered the cohomology of the corresponding *pro-motives* as it was done in §4 of [Deg08]; this wouldn't have affected the proof substantially. Unfortunately, the category of pro-motives is not triangulated (if we define it in the obvious way).

A certain triangulated analogue of pro-motives (a category of *comotives*) was constructed in [Bon10] (see §1.5, §3.1, and §5 of *ibid.*). It contains more information, yet is somewhat more difficult to deal with (in particular, the author currently does not know how to control the corresponding 'homotopy limits' of motives unless k is countable). In this category the general weight structure formalism can be applied directly; this allows to get rid off the codimension condition for d_i . Unfortunately, it seems difficult to describe the corresponding weight decompositions of motives and their morphisms in the general case; one only knows that they can be constructed from coniveau filtration of U_r (as in Lemma 1.5.4).

5. As was noted in (part 6 of) Remark 4.4.3, all these statements could be vastly generalized.

7.5 Expressing torsion motivic cohomology (with compact support) in terms of étale one

For fixed $n > 0$, $(l, p) = 1$, $r \geq 0$, we denote by $\mathcal{H}_{l^n}^{et}(r) \in \text{Obj} D^-(\text{Shv}(\text{SmCor}))$ some étale resolution of $\mu_{l^n}^{\otimes r}$ by injective étale sheaves with transfers. This object does not depend on the choice of a resolution (as an object of the derived category) by obvious reasons (it is the total derived image of $\mu_{l^n}^{\otimes r}$ with respect to the corresponding change of topologies functor). $\mathcal{H}_{l^n}^{et}(r)$ is homotopy invariant, so we can substitute it for H in the statements above (since for any fixed M it suffices to consider some homotopy t -truncation of $\mathcal{H}_{l^n}^{et}(r)$, whereas the latter belong to $\text{Obj} DM_-^{eff}$).

Proposition 7.5.1 (The Beilinson-Lichtenbaum Conjecture). *The (well-known) cycle class map $\mathbb{Z}/l^n\mathbb{Z}(r) \rightarrow \mathcal{H}_{l^n}^{et}(r)$ identifies the former object with $\tau_{\leq r}\mathcal{H}_{l^n}^{et}(r)$.*

We recall that this statement is equivalent to the Bloch-Kato Conjecture (see [SuV00] and [Gel01]). The latter is known for $l = 2$ (see [Voe03]); so reader may assume that the next result is stated for $l = 2$. For odd l it was recently announced (by Rost, Voevodsky, and Weibel) that the missing details for the famous Voevodsky's plan of the proof of the conjecture are completed.

For a motif X we denote $DM_-^{eff}(X, \mathbb{Z}/l^n\mathbb{Z}(s)[i])$ by $H^i(X, \mathbb{Z}/l^n\mathbb{Z}(s))$; $H_{et}^i(X, \mathbb{Z}/l^n\mathbb{Z}(s)) = D^-(\text{Shv}(\text{SmCor}))(X, \mathcal{H}_{l^n}^{et}(s)[i])$.

Now, Theorem 7.4.2 easily yields the following statement (in the notation of *loc.cit.*).

Corollary 7.5.2. 1. For M as in (32), we have

$$H^j(M, \mathbb{Z}/l^n\mathbb{Z}(s)) \cong \text{Im}(\varinjlim H_{et}^j(N^0, \mathbb{Z}/l^n\mathbb{Z}(s)) \rightarrow \varinjlim H_{et}^j(N^1, \mathbb{Z}/l^n\mathbb{Z}(s))). \quad (39)$$

The corresponding functor H_s'' can be calculated as follows:

$$H_s''^j(M) \cong \text{Im}(\varinjlim H_{et}^j(L^0, \mathbb{Z}/l^n\mathbb{Z}(s)) \rightarrow \varinjlim H_{et}^j(L^1, \mathbb{Z}/l^n\mathbb{Z}(s))).$$

These isomorphisms satisfy those functoriality properties that were described in part III of Theorem 7.4.2.

2. Let Y_h , $1 \leq h \leq u$, $u > 0$, be smooth of the same dimension; let $Y = \cup Y_h$ be a normal crossing scheme i.e. all intersections (in some large basic scheme) are normal and smooth. Consider the motif M corresponding to the complex (U_l) ; here $U_l = \sqcup_{(i_j)} Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_{l+1}}$ for all $1 \leq i_1 \leq \cdots \leq i_{r+1} \leq u$, d_l is the alternated sum of $l+1$ natural maps $U_l \rightarrow U_{l-1}$.

Let (Y^0, Y^1) run through open subschemes of Y such that $Y \setminus Y_r$ is (everywhere) of codimension $\geq j - r - s + 1$ in Y ($r = 0, 1$). Then we have

$$H_s''^j(M) \cong \text{Im}(\varinjlim H_{et}^j(Y^0, \mathbb{Z}/l^n\mathbb{Z}(s)) \rightarrow \varinjlim H_{et}^j(Y^1, \mathbb{Z}/l^n\mathbb{Z}(s))). \quad (40)$$

For $N^r = M_{gm}(Y^r \rightarrow Y)$ (Y is in degree 0) we have

$$H^j(M, \mathbb{Z}/l^n\mathbb{Z}(s)) \cong \text{Im}(\varinjlim H_{et}^j(N^0, \mathbb{Z}/l^n\mathbb{Z}(s)) \rightarrow \varinjlim (H_{et}^j(N^1, \mathbb{Z}/l^n\mathbb{Z}(s))). \quad (41)$$

3. Let $Y' = \cup Y'_i$, M' is defined similarly to M , let $f : Y' \rightarrow Y$ be a morphism of schemes, suppose that for any (closed) point $u \in Y$ we have $\text{codim}_{Y'} f^{-1}(u) \geq \text{codim}_Y u - 1$. Then the morphisms $f_{H'_s}^*$ and $f_{H''_s}^*$ can be computed by the way described in part III2 of Theorem 7.4.2 (see also its proof and part 2 of Remark 7.4.3).

4. Let $U \in \text{SmVar}$ equal $P \setminus Y = \cup Y_i$, $1 \leq i \leq m$, where $P \in \text{SmPrVar}$, Y is a smooth normal crossing divisor. Then for any cohomological functor G defined on DM_-^{eff} we have a long exact sequence

$$\cdots G^j(M_{gm}^c(U)) \rightarrow G^j(X) \rightarrow G^j(M) \rightarrow \cdots \quad (42)$$

Proof. 1. This is immediate from part 2 of Theorem 7.4.2 applied for $H = \mathcal{H}_{l^n}^{et}(s)$, $i = s$.

2. We should prove that the formulas (40) and (41) compute the limits described in parts 1,2 of Theorem 7.4.2.

First, we note that if Y^r ($r = 0, 1$) satisfies the condition of the assertion then

$$Y_l^r = \sqcup_{(i_j)} Y^r \cap Y_{i_1} \cap Y_{i_2} \cap \cdots \cap Y_{i_{l+1}} \quad (43)$$

satisfies the conditions of part 1 of Theorem 7.4.2. Next, using proper descent we easily obtain that the étale cohomology of Y^r is isomorphic to those of $L^r = (Y_l^r)$. It suffices to note that any set of Y_l^r as in part 1 of Theorem 7.4.2 could be shrunk to a one coming from some Y^r as in (43).

3. It suffices to note that the functoriality provided by part III2 of Theorem 7.4.2 is compatible with those of the formulas (40) and (41).

4. By definition, $H_c^j(U, \mathbb{Z}/l^n\mathbb{Z}(s)) = H^j(M_{gm}^c(U), \mathbb{Z}/l^n\mathbb{Z}(s))$. Hence it suffices to recall that $M_{gm}^c(U) \cong \text{Cone}(M \rightarrow M_{gm}(X))$. The latter fact in the characteristic 0 case is Proposition 6.5.1 of [Bon09]. In the characteristic p case one could deduce the statement from the results of [Del71] (and the Poincaré duality properties).

□

Remark 7.5.3. 1. Note that for $G = H'$ or $G = H''$ one could compute the map $G^*(M) \rightarrow G^*(Y)$ in (42) using part 3 of the corollary.

Besides, one can write down the formula for the (motivic and H'' -) cohomology of $M_{gm}^c(U)$ by substituting the 'complex' $Y \rightarrow X$ for M into part 1 of the corollary; here one should ignore the fact that Y could be singular.

2. Let K contain a primitive l^n -th root of unity. Then using it one can identify all $\mathcal{H}_{l^n}^{et}(r)$, and so obtain certain maps $H^i(-, \mathbb{Z}/l^n\mathbb{Z}(s)) \rightarrow H^{i+j}(-, \mathbb{Z}/l^n\mathbb{Z}(s+j))$ (induced by the multiplication on the corresponding motivic Bott elements, as in [Lev00]). Then (38) allows to calculate the image of these maps.

One could prove natural analogues of part 2 of the corollary and part 1 of this remark.

3. It seems very interesting to replace étale cohomology in the right hand side of (41) by singular cohomology (in the case when k is the field of complex numbers). One could easily deduce from Corollary 7.4 of [BOg94] that in the case $u = 0$, $j = 2s$, the formula would calculate the group of algebraic cycles in U_0 of codimension s modulo algebraic equivalence. So it seems that the homotopy t -structure truncations of singular cohomology should be related to a certain (non-existent yet) 'theory of mixed motives up to algebraic equivalence'; see the end of [Voe95].

4. Certainly, the cohomology of M is a very natural candidate for the cohomology of Y ; note that $\cup Y_j \rightarrow Y$ is a *cdh*-covering (see [FrV00]). Yet this does not automatically imply the isomorphism of cohomology for all 'reasonable' cohomology theories.

5. Recall that if k admits resolution of singularities any smooth quasi-projective U can be presented as $X \setminus \cup Y_i$.

7.6 The cases when t_{Chow} can be easily calculated; relation with unramified cohomology

By Theorem 4.4.2, one can express the values of $DM_-^{eff}(X, t_{Chow}^{\leq i}(Y)[j])$ (for example, in the case $X \in Obj DM_{gm}^{eff}$, $Y \in Obj DM_-^{eff}$) in terms of Y -cohomology of weight truncations of X . It turns out that in some cases one obtains very nice results this way; these are related with unramified cohomology (see Remark 7.6.2) below.

Proposition 7.6.1. *Let $U \in SmVar$, let P be any smooth compactification of U (i.e. U is open in P , P is smooth proper).*

1. *Let $\text{char } k = 0$. Then for any homotopy invariant $S \in Shv(SmCor)$ we have $\underline{C}(M_{gm}(U), S^{t_{Chow}=0}) \cong S(P)$.*

2. *The same is true for any perfect k if S is a sheaf of \mathbb{Q} -vector spaces.*

Proof. For the proof of part 1 we take $\underline{C} = DM_-^{eff}$, for part 2 $\underline{C} = DM_{\mathbb{Q}}^{eff}$. We will use the same notation for motives of varieties in these categories, and for their weight and t -structures.

We have morphisms $M_{gm}(U) \xrightarrow{g} w_{\leq 1}M_{gm}(U) \xrightarrow{f} w_{\leq 0}M_{gm}(U)$. By part 8 of Theorem 4.4.2, we have $\underline{C}(M_{gm}(U), t_{Chow}^{\leq 0}(S)) = \text{Im } f^*(S)$.

Now, by Lemma 7.6.3 below, we can assume that $w_{\leq 0}(M_{gm}(U)) = M_{gm}(P)$. For this choice $(f \circ g)_S^*$ is injective by Corollary 4.19 of [Voe00b]. Hence f_S^* is injective also, and we get $S(P) \cong t_{Chow}^{\leq 0}(S)$.

Lastly, we note that S belongs to $\underline{C}^{t_{Chow} \leq 0}$ (since smooth proper varieties have no negative S -cohomology; note that the construction of t_{Chow} uses Theorem 4.5.2). Hence $t_{Chow}^{\leq 0}(S) = S^{t_{Chow}=0}$. □

Remark 7.6.2. 1. The statement proved immediately yields: $S(P)$ is a birational invariant of P (i.e. it depends only on the function field of P); cf. Theorem 8.5.1 of [C-THK].

Now we relate the statement proved with unramified cohomology (as defined in §4.1 of [Co-T95]). Suppose that a cohomology theory can be represented by $C \in Obj \underline{C}$ (or of the unbounded version of DM_-^{eff}); for example, this is (essentially) the case for torsion étale cohomology and de Rham cohomology. Then for $U \in SmVar$ the i -th unramified cohomology of $k(U)$ (the function field) with coefficients in C equals $S^i(P)$, where $S^i = C^{t=i}$, P is a smooth compactification of U . A similar statement was verified in Theorem 4.1.1 of [Co-T95]; note that the results of [Voe00b], §4, yield all properties of $\underline{C}(-, C)$ that are necessary for the proof.

2. Now, let $C \in \underline{C}^{t \geq 0}$ (with rational coefficients if $\text{char } k > 0$). Certainly, one has $\underline{C}(M_{gm}(U), t_{Chow}^{\leq 0} C) = \underline{C}(M_{gm}(P), C)$. Indeed, it suffices to apply the Proposition above for $S = H^{t=0}(C)$.

More generally, by analyzing weight decompositions of $M_{gm}(U)$ in more detail (and fixing their choice), one can check that for any $i \geq 0$, $C \in \underline{C}^{t \geq -2i}$ the map $\underline{C}(w_{\leq i} M_{gm}(U), C) \rightarrow \underline{C}(w_{\leq i+1} M_{gm}(U), C)$ is injective. Therefore, we have $\underline{C}(M_{gm}(U), C^{t_{Chow} \geq -i}) = \underline{C}(w_{\leq i} M_{gm}(U), C)$ (for a 'nice' choice of $\underline{C}(w_{\leq i} M_{gm}(U))$).

Moreover, one could replace the functor $\underline{C}(-, C)$ by any cohomological $H : \underline{C} \rightarrow A$ (A is an abelian category; in the case $\text{char } k > 0$ it should be \mathbb{Q} -linear) that satisfies: $H(M_{gm}(U)[i]) = 0$ for any $U \in SmVar$, $i < 0$. Then the statements above (of part 2 of this Remark) will also be true if one defines $H^{t_{Chow} \geq -i}$ using virtual t -truncations (see Remark 2.5.2).

3. As mentioned in Remark 6.6.1, the author hopes to extend the results of this section to the setting of $\mathbb{Z}[1/p]$ -coefficients (in the case $\text{char } k = p$).

Now we prove the statement that was used in the proof of Proposition 7.6.1

Lemma 7.6.3. *In the conditions of Proposition 7.6.1 the map $M_{gm}(U) \rightarrow M_{gm}(P)$ can be extended to a weight decomposition of $M_{gm}(U)$.*

Proof. We should prove that $\text{Cone}(M_{gm}(U) \rightarrow M_{gm}(P)) \in \underline{C}^{w \geq 1}$. In the case $U = P \setminus Z$, $Z \subset P$ is smooth projective, this statement is immediate from the Gysin distinguished triangle $M_{gm}(U) \rightarrow M_{gm}(P) \rightarrow M_{gm}(Z_i)(r_i)[2r_i]$, Z_i are connected components of Z , r_i are their codimensions (see Proposition 3.5.4 of [Voe00a] for $\text{char } k = 0$ and Proposition 5.21 of [Deg08] for the general case).

In the general case the statement is also proved by induction. We choose a stratification of $P \setminus U = \cup Z_i$, where $Z_i \setminus Z_{i+1} \in SmVar$, $Z_m = \{0\}$ for some m . Then we can apply the Gysin triangle for the pair $(P \setminus Z_{i+1}, Z_i \setminus Z_{i+1})$. Part 1 of Remark 6.5.2 yields that each of $M_{gm}(Z_i \setminus Z_{i+1})$ belongs to $\underline{C}^{w \geq 0}$. Note here that the latter statement is true for motives with rational coefficients in any characteristic, since alterations yield that the motif of any smooth variety can be presented as a retract of a motif of a complement of a smooth normal crossing divisor; see Appendix B of [HuK06].

□

8 Supplements

We start the section by proving that weight structures have nice 'functorial' properties similar to those of t -structures (yet the difference is substantial).

In §8.1 we show that a weight structure w on \underline{C} which induces a weight structure on a triangulated $\underline{D} \subset \underline{C}$ yields also a weight structure on the localization $\underline{C}/\underline{D}$. Note that the heart of the weight structure for $\underline{C}/\underline{D}$ has a simple description in terms of $\underline{Hw}_{\underline{C}}$ and $\underline{Hw}_{\underline{D}}$; yet this description is quite distinct from the corresponding one for hearts of t -structures.

In §8.2 we prove a certain converse to this statement: weight structures could be 'glued' in a manner that is similar to those for t -structures. The author expects to apply this fact for the construction of weight structures for relative motives (and thus, weights for their realizations).

In §8.3 we study the interaction of weight and t -structures. Let a triangulated category be endowed with several (adjacent, distinct) weight and t -structures. We show that functors represented by compositions of t -truncations (with respect to possibly distinct t -structures) could be expressed in terms of the corresponding adjacent weight structures; the formulae are similar to those of Theorem 4.4.2 (and are deduced from it). We also extend this result to the case of several triangulated categories connected by exact functors.

In §8.4 we prove (using an argument due to A. Beilinson) that any f -category enhancement of \underline{C} yields a lift of t to a 'strong' weight complex functor $\underline{C} \rightarrow K(\underline{Hw})$; cf. Remark 3.3.4.

In §8.5 we discuss other possible sources of conservative 'weight complex-like' functors and related spectral sequences.

We conclude the section by a discussion of three relevant types of filtrations for triangulated categories; all of them 'should' be actual (and closely related) for $DM_{gm}^{eff}\mathbb{Q}$.

8.1 Weight structures in localizations

We call a category $\frac{A}{B}$ a *factor* of an additive category A by its full additive subcategory B if $Obj(\frac{A}{B}) = Obj A$ and $(\frac{A}{B})(X, Y) = A(X, Y) / (\sum_{Z \in Obj B} A(Z, Y) \circ A(X, Z))$.

Proposition 8.1.1. *1. Let $\underline{D} \subset \underline{C}$ be a triangulated subcategory of \underline{C} ; suppose that w induces a weight structure on \underline{D} (i.e. $Obj \underline{D} \cap \underline{C}^{w \leq 0}$ and $Obj \underline{D} \cap \underline{C}^{w \geq 0}$ give a weight structure for \underline{D}). We denote the heart of the latter weight structure by HD .*

Then w induces a weight structure on $\underline{C}/\underline{D}$ (the localization i.e. the Verdier quotient of \underline{C} by \underline{D}). This means that the Karoubi-closures of $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ (in $\underline{C}/\underline{D}$) give a weight structure for $\underline{C}/\underline{D}$ (note that $Obj \underline{C} = Obj \underline{C}/\underline{D}$).

2. $H(\underline{C}/\underline{D})$ is the Karoubi-closure of $\frac{Hw}{HD}$ in $\underline{C}/\underline{D}$.

3. If \underline{C}, w is bounded (above, below, or both), then $\underline{C}/\underline{D}$ also is.

Proof. 1. It clearly suffices to prove that for any $X \in (\underline{C}/\underline{D})^{w \geq 0}$ and $Y \in (\underline{C}/\underline{D})^{w \leq -1}$ we have $(\underline{C}/\underline{D})(X, Y) = \{0\}$; all other axioms of Definition 1.1.1 are fulfilled automatically since $\underline{C}/\underline{D}$ is a localization of \underline{C} .

Recall now (see Lemma III.2.8 of [GeM03]) that any morphism in $(\underline{C}/\underline{D})(X, Y)$ can be presented as fs^{-1} where $f \in \underline{C}(T, Y)$ for some $T \in \text{Obj}\underline{C}$, $s \in \underline{C}(T, X)$, $\text{Cone}(s) = Z \in \text{Obj}\underline{D}$.

By our assertion, there exists a choice of $Z^{w \geq 0}$ that belongs to $\text{Obj}\underline{D}$. Since $X \perp w_{\leq -1}Z$ we can factorize the morphism $X \rightarrow Z$ (induced by s) through $Z^{w \geq 0}$.

Hence (applying the octahedral axiom) we obtain that there exist $T' \in \text{Obj}\underline{C}$, a morphism $d : T' \rightarrow T$, such that $\text{Cone } d = w_{\leq -1}Z \in \text{Obj}\underline{D}$, whereas a cone of the composite morphism $s' : T' \rightarrow X$ equals $Z^{w \geq 0}$. It follows that $fs^{-1} = (fd)s'^{-1}$ in $\underline{C}/\underline{D}$. Now note that $T' \in \underline{C}^{w \geq 0}$ by part 3 of Proposition 1.3.3. Hence $T \perp Y$, which yields $fd = 0$.

2. By construction, $\underline{C}^{w=0} \subset (\underline{C}/\underline{D})^{w=0}$.

Now we prove that any object of $H(\underline{C}/\underline{D})$ is a retract of an object of \underline{Hw} (in $\underline{C}/\underline{D}$).

Let $Z \in \underline{C}/\underline{D}^{w=0} \subset \text{Obj}\underline{C}$. We consider a weight decomposition $w_{\geq 1}(Z) \rightarrow Z \rightarrow w_{\leq 0}Z$ of Z in \underline{C} . In $\underline{C}/\underline{D}$ we have $w_{\geq 1}(Z) \in \underline{C}/\underline{D}^{w \geq 1}$, hence $\underline{C}/\underline{D}(w_{\geq 1}(Z), Z) = \{0\}$. Therefore Z in $\underline{C}/\underline{D}$ is a retract of $Z^{w \leq 0}$. Moreover, $Z^{w \geq 1} \in \underline{C}/\underline{D}^{w=0}$ since it is a retract of $Z^{w \leq 0} \in \underline{C}/\underline{D}^{w \leq 0}$; therefore $Z^{w \leq 0} \in \underline{C}/\underline{D}^{w=0}$. Now applying the dual argument to $Z^{w \leq 0}$ (see Remark 1.1.2), we obtain that Z in $\underline{C}/\underline{D}$ is a retract of some $Z^0 \in \text{Obj}\underline{C}^{w=0}$.

To conclude the proof it suffice to check that the natural functor $i : \underline{Hw}/\underline{HD} \rightarrow H(\underline{C}/\underline{D})$ is a full embedding. We consider the composition $\underline{C} \xrightarrow{t} K_{\mathfrak{w}}(\underline{Hw}) \rightarrow K_{\mathfrak{w}}(\underline{Hw}/\underline{HD})$. Obviously, it maps all objects of \underline{D} to 0. Hence i is injective on morphisms.

It remains to prove that any morphism $g : X \rightarrow Y$ in $\underline{C}/\underline{D}$ comes from $\underline{C}(X, Y)$. Applying the same argument as in the proof of assertion 1 we obtain that g can be presented as fs^{-1} where $f \in \underline{C}(T, Y)$ for some $T \in \text{Obj}\underline{C}$, $s \in \underline{C}(T, X)$, $\text{Cone } s = Z \in \underline{D}^{w \geq 0}$. Then $\underline{C}(X, Y)$ surjects onto $\underline{C}(T, Y)$. Now the 'calculus of fractions' yields the result.

3. Since $\text{Obj}\underline{C}/\underline{D} = \text{Obj}\underline{C}$, we obtain the claim. □

Corollary 8.1.2. *Let $E \subset \underline{Hw}$ be an additive subcategory. If X belongs to the Karoubi-closure $\text{Obj}\langle E \rangle$, then $t(X)$ is a retract of some object of $K_{\mathfrak{w}}^b(E)$ (here we assume that $K_{\mathfrak{w}}^b(E) \subset K_{\mathfrak{w}}(HC)$).*

If (\underline{C}, w) is bounded then the converse implication also holds.

Proof. We can assume that $X \in \text{Obj}\langle E \rangle$. Then X can be obtained from objects of E by repetitive consideration of cones of morphisms. Since $t(\text{Obj} E) \subset$

$ObjK_{\mathfrak{w}}(E)$ and t is a weakly exact functor in the sense of Definition 3.1.6, we obtain that $t(X) \in ObjK_{\mathfrak{w}}^b(E)$.

Conversely, let $t(X)$ be a retract of $Y \in ObjK_{\mathfrak{w}}^b(E) \subset ObjK_{\mathfrak{w}}^b(\underline{Hw})$. By Proposition 8.1.1 we obtain that $\underline{C}/\langle E \rangle$ possesses a bounded weight structure whose heart contains $\frac{Hw}{E}$ as a full subcategory. Hence, by part V of Theorem 3.3.1 we obtain that $t_{\underline{C}/\langle E \rangle}$ is conservative. $Y \in ObjK_{\mathfrak{w}}^b(E)$ gives $t_{\underline{C}/\langle E \rangle}(Y) = 0$, hence X and Y belong to the Karoubi-closure of $\langle E \rangle$. \square

Remark 8.1.3. 1. Note that (in general) one cannot be sure that the 'factor weight structure' on $\underline{C}/\underline{D}$ is non-degenerate.

2. Corollary 8.1.2 is parallel to part 3 of Proposition 8.2.1 of [Bon09]. In particular, it could be used to prove that a motif of a smooth variety is mixed Tate whenever its weight complex (defined in [GiS96]) is (this is Corollary 8.2.2 of [Bon09]).

3. Adding certain additional restrictions, one could also formulate a criterion for $t(X)$ to belong to the Karoubi-closure of $ObjK_{\mathfrak{w}}(E)$ (instead of $ObjK_{\mathfrak{w}}^b(E)$).

4. One can (easily) apply Proposition 8.1.1 for the calculation of $\text{Hom}(\bar{M}(X), \bar{M}(Y)[i])$ for $i \geq 0$; see Corollary 7.9 of [KaS02] (note that the latter statement is false for $i < 0$). Here $X, Y \in SmPrVar$, $\bar{M}(X), \bar{M}(Y)$ are their birational motives considered as objects of the triangulated category of birational motives (see §5 of [KaS02]).

5. A terminological comment: under the assumptions of Proposition 8.1.1 the inclusion $\underline{D} \rightarrow \underline{C}$ and the projection $\underline{C} \rightarrow \underline{C}/\underline{D}$ are weight-exact (in the sense of Definition 4.4.4).

8.2 Gluing weight structures; weights for relative motives

Since weight structures are often 'dual' to t -structures (see §4), this is no surprise that one can modify the 'gluing' procedure of §1.4 of [BBD82] (for t -structures) so that it can be applied to weight structures (yet cf. part 4 of Remark 8.2.4 below).

Our result is a certain converse to the result of the previous subsection: we study when a weight structure for \underline{C} can be recovered from weight structures on a triangulated $\underline{D} \subset \underline{C}$ and on the localization of \underline{C} by \underline{D} . As in the similar situation for t -structures, we need *gluing data* i.e. certain adjoint functors should exist.

We describe gluing data for abstract triangulated categories as it was done in §1.4.3 of [BBD82] (we will only change the notation for categories; see also

the exercises at the end of §IV.4 of [GeM03]). Still recall that usually gluing data sets come from certain derived categories of sheaves; this also explains our notation for functors (yet note that we are actually interested in the derived versions of the corresponding functors on categories of sheaves). In particular, the category \underline{D} (below) usually comes from a closed subspace of the space corresponding to \underline{C} , whereas \underline{E} comes from its (open) complement.

It is well known (see Chapter 9 of [Nee01]) that gluing data can be uniquely recovered from an inclusion $\underline{D} \rightarrow \underline{C}$ of triangulated categories that admits both a left and a right adjoint functor. Then for \underline{E} being the Verdier quotient $\underline{C}/\underline{D}$ the projection $\underline{C} \rightarrow \underline{E}$ also admits both a left and a right adjoint. Following [BBD82], we summarize the properties of the functors obtained (and introduce notation for them).

Definition 8.2.1. The set $(\underline{C}, \underline{D}, \underline{E}, i_*, j^*, i^!, j_!, j_*)$ is called *gluing data* if it satisfies the following conditions.

(i) $\underline{C}, \underline{D}, \underline{E}$ are triangulated categories; $i_* : \underline{D} \rightarrow \underline{C}$, $j^* : \underline{C} \rightarrow \underline{E}$, $i^* : \underline{C} \rightarrow \underline{D}$, $i^! : \underline{C} \rightarrow \underline{D}$, $j_* : \underline{E} \rightarrow \underline{C}$, $j_! : \underline{E} \rightarrow \underline{C}$ are exact functors.

(ii) i^* (resp. $i^!$) is left (resp. right) adjoint to i_* ; $j_!$ (resp. j_*) is left (resp. right) adjoint to j^* .

(iii) i_* is a full embedding; j^* is isomorphic to the localization (functor) of \underline{C} by $i_*(\underline{D})$.

(iv) For any $X \in \text{Obj} \underline{C}$ the pairs of morphisms $j_! j^* X \rightarrow X \rightarrow i_* i^* X$ and $i_* i^! X \rightarrow X \rightarrow j_* j^* X$ can be completed to distinguished triangles (here the connecting morphisms come from the adjunctions of (ii)).

(v) $i^* j_! = 0$; $i^! j_* = 0$.

(vi) All of the adjunction transformations $i^* i_* \rightarrow id_{\underline{D}} \rightarrow i^! i_*$ and $j^* j_* \rightarrow id_{\underline{E}} \rightarrow j^* j_!$ are isomorphisms of functors.

Now suppose that $\underline{D}, \underline{E}$ are endowed with weight structures, that we will (by an abuse of notation) denote by w . We prove that there exists a (unique) weight structure for \underline{C} such that i_* and j^* are weight-exact (with respect to weight structures mentioned; see Definition 4.4.4). To this end we consider $\underline{C}^{w \leq 0} = \{X \in \text{Obj} \underline{C} : i^! X \in \underline{D}^{w \leq 0}, j^* X \in \underline{E}^{w \leq 0}\}$ and $\underline{C}^{w \geq 0} = \{X \in \text{Obj} \underline{C} : i^* X \in \underline{D}^{w \geq 0}, j_* X \in \underline{E}^{w \geq 0}\}$.

Before proving that these classes actually define a weight structure, we will study how they behave with respect to the connecting functors.

Lemma 8.2.2. 1. j_* maps $\underline{E}^{w \leq 0}$ to $\underline{C}^{w \leq 0}$.

2. $j_!$ maps $\underline{E}^{w \geq 0}$ to $\underline{C}^{w \geq 0}$.

3. i_* maps $\underline{D}^{w \leq 0}$ (resp. $\underline{D}^{w \geq 0}$, resp $\underline{D}^{w=0}$) to $\underline{C}^{w \leq 0}$ (resp. to $\underline{C}^{w \geq 0}$, resp to $\underline{C}^{w=0}$).

Proof. 1. As we know, for any $Z \in \text{Obj } \underline{E}$ we have $j^*j_*(Z) \cong Z$, $i^!j_*Z = 0$. Hence for $Z \in \text{Obj } \underline{E}^{\leq 0}$ we have $j^*j_*(Z) \in \underline{E}^{w \leq 0}$, $i^!j_*Z \in \underline{D}^{w \leq 0}$.

2. Similarly to the proof of part 1, it suffices to note that for any $Z \in \text{Obj } \underline{E}$ we have $j^*j_!(Z) \cong Z$, $i^*j_*Z = 0$.

3. For any $Y \in \text{Obj } \underline{D}$ we have $i^!i_*(Y) \cong i^*i_*Y \cong Y$, $j^*i_*Y = 0$. Hence we obtain the statement for $\underline{D}^{w \leq 0}$ and $\underline{D}^{w \geq 0}$ (similarly to part 1). Since $\underline{D}^{w=0} = \underline{D}^{w \leq 0} \cap \underline{D}^{w \geq 0}$, the last part of the assertion follows from the previous ones immediately. \square

Theorem 8.2.3. *The classes $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ define a weight structure for \underline{C} .*

Proof. The definitions of $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ immediately yield that they are Karoubi-closed and semi-invariant with respect to translations (in the sense of Definition 1.1.1). They are also extension-stable (see Definition 1.3.1).

The proof of orthogonality is similar to those in Theorem 1.4.10 of [BBD82]. Let $X \in \text{Obj } \underline{C}^{w \geq 1}$, $X' \in \underline{C}^{w \leq 0}$. We should prove that $X \perp X'$.

Part (iv) of Definition 8.2.1 yields a (long) exact sequence $\cdots \rightarrow \underline{C}(i_*i^*X, X') \rightarrow \underline{C}(X, X') \rightarrow \underline{C}(j_*j^*X, X') \rightarrow \dots$. It remains to note that the adjunctions of functors (of part (ii) of Definition 8.2.1), and the definitions of $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ yield:

$$\underline{C}(i_*i^*X, X') = \underline{D}(i^*X, i^!X') = \{0\} = \underline{E}(j^*X, j^*X') = \underline{C}(j_*j^*X, X').$$

The last axiom check is the existence of weight decompositions. By Remark 1.5.5, it suffices to check that a generating family of objects of \underline{C} possess weight decompositions (into objects belonging to our $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$). Now, part 3 of Lemma 8.2.2 immediately yields that for any object of $i_*(\underline{D})$ a weight decomposition exists (since one can take those weight decompositions that come from \underline{D} via i_*). Also, by parts 1,2 of the lemma, one could take 'trivial' weight decompositions for objects of $j_*(\underline{E}^{w \leq 0})$ and $j_!(\underline{E}^{w \geq 1})$.

Next, by the definition of weight structures and part (v) of Definition 8.2.1, the objects of $j^*j_*(\underline{E}^{w \leq 0})$, and $j^*j_!(\underline{E}^{w \geq 0})$ generate \underline{E} (together) as a triangulated category. Hence part (iii) of Definition 8.2.1 yields that the objects of $i_*(\underline{D})$, $j_*(\underline{E}^{w \leq 0})$, and $j_!(\underline{E}^{w \geq 0})$ generate \underline{C} as a triangulated category. The theorem is proved. \square

Remark 8.2.4. 1. The argument used for the proof of existence of weight decompositions also yields: $\underline{C}^{w \leq 0}$ equals the Karoubi-closure of the smallest extension-stable subclass of $\text{Obj } \underline{C}$ containing $\text{Obj } j_*(\underline{E}^{w \leq 0}) \cup \text{Obj } j_!(\underline{D}^{w \leq 0})$ (cf.

the proof of part III of Theorem 4.3.2); $\underline{C}^{w \geq 0}$ equals the (similarly defined) 'envelope' of $Obj j_!(\underline{E}^{w \geq 0}) \cup Obj j_*(\underline{D}^{w \geq 0})$.

2. Using adjunctions and parts 1,2 of Proposition 1.3.3 we obtain: $X \in \underline{C}^{w \leq 0}$ (resp. $X \in \underline{C}^{w \geq 0}$) whenever $j^*X \in \underline{E}^{w \leq 0}$ and for any $Y \in \underline{D}^{w \geq 1}$ we have $i_*Y \perp X$ (resp. $j^*X \in \underline{E}^{w \geq 0}$ and for any $Y \in \underline{D}^{w \leq -1}$ we have $X \perp i_*Y$).

It follows that in order to 'calculate' w for \underline{C} , it suffices to know only i_* and j^* . Note that these two functors could be easier to describe than the remaining ones (in particular, for the categories of relative motives; see below).

3. We plan to construct a Chow weight structure for the (triangulated) category DM_S of relative motives i.e. of motives over a base scheme S that is not a field. We will probably treat this subject in detail in a forthcoming paper.

One of the possible methods for this is to use gluing. By simplicity, we will assume here that S is a scheme over \mathbb{Q} , and give a rough sketch of the construction of w_{Chow} for DM_S in this case (this method works better if we consider the categories of so-called 'constructible' motives, though the Chow weight structure could also be extended to the 'big' category of Voevodsky's motives over S). First one should prove (using certain Poincare duality): there exists a weight structure on the subcategory $DM_{pr,S}$ of motives over S generated by motives of smooth projective schemes over S . Yet there is a problem here: $DM_{pr,S}$ is usually much smaller than DM_S .

The proof goes on by Noetherian induction. We can construct the Chow weight structure on DM_S (for a scheme S/k which is not necessarily smooth) assuming that for all closed $T \subset S$, $T \neq S$, the Chow weight structures for DM_T are already defined and all of these structures are compatible with each other. Then one takes for \underline{E} the category of motives over an arbitrary fixed generic point of S . We define triangulated categories D_l as DM_{T_l} , where T_l , $l \in L$, runs through all closed irreducible proper subschemes of S . Then one can obtain a weight structure on DM_S by using almost the same construction as above: $\underline{C}^{w \leq 0} = \{X \in Obj \underline{C} : i_l^!X \in \underline{D}_l^{w \leq 0} \forall l \in L, j^*X \in \underline{E}^{w \leq 0}\}$ and $\underline{C}^{w \geq 0} = \{X \in Obj \underline{C} : i_l^*X \in \underline{D}_l^{w \geq 0} \forall l \in L, j^*X \in \underline{E}^{w \geq 0}\}$.

As in the proof of Theorem 8.2.3, $\underline{C}^{w \leq 0}$ and $\underline{C}^{w \geq 0}$ are obviously Karoubi-closed and semi-invariant with respect to translations. Using the theorem, one can reduce the axioms of weight structures check to the case of an irreducible S (so, it has a unique generic point). The main property needed to conclude the proof (in this case) is that $\underline{E} = \varinjlim DM_{S \setminus T_l}$; this immediately implies $\underline{E} = \varinjlim DM_{pr,S \setminus T_l}$. Moreover, we have $\underline{E}^{w \leq 0} = \varinjlim DM_{pr,S \setminus T_l}^{w \leq 0}$, $\underline{E}^{w \geq 0} = \varinjlim DM_{pr,S \setminus T_l}^{w \geq 0}$.

Since S is a \mathbb{Q} -scheme, for any $X \in Obj DM_S$ there exists an open $U \subset S$

such that the 'restriction' of X to DM_U belongs to $DM_{pr,U}$. Moreover, if $j^*X \in E^{w \leq 0}$ (resp. $j^*X \in E^{w \geq 0}$) then the 'restriction' of X to some open U belongs to $DM_{pr,U}^{w \leq 0}$ (resp. $DM_{pr,U}^{w \geq 0}$). Then one can prove the orthogonality property for $X \in Obj \underline{C}^{w \geq 1}$, $X' \in \underline{C}^{w \leq 0}$, by choosing a common U for them and then applying the corresponding reasoning used in the proof of Theorem 8.2.3 (for the case $\underline{E} = DM_{pr,U}$, $\underline{D} = DM_{S \setminus U}$). Lastly, in order to obtain a weight decomposition for X it suffices to decompose a 'restriction' of it that belongs to $DM_{pr,U}$ for some U and also certain elements of $Obj DM_{S \setminus U}$ (see the argument used to prove the existence of weight decompositions in the theorem).

Certainly, a success in this program would immediately yield weights for arbitrary cohomology of relative motives.

4. To the surprise of the author, it seems that one cannot glue adjacent weight and t -structures in a compatible way (in the general case). Indeed, suppose that one has t -structures on \underline{D} and \underline{E} that are left adjacent to the corresponding weight structures. Following Theorem 1.4.10 of [BBD82], one should define t on \underline{C} as follows: $\underline{C}^{t \leq 0} = \{X \in Obj \underline{C} : i^*X \in \underline{D}^{t \leq 0}, j^*X \in \underline{E}^{t \leq 0}\}$ and $\underline{C}^{t \geq 0} = \{X \in Obj \underline{C} : i^!X \in \underline{D}^{t \geq 0}, j^*X \in \underline{E}^{t \geq 0}\}$. It follows that $j_!$ maps $\underline{E}^{t \leq 0}$ to $\underline{C}^{t \leq 0}$; j_* maps $\underline{E}^{t \geq 0}$ to $\underline{C}^{t \geq 0}$. Now, suppose that w is left adjacent to t on \underline{C} . In order to formulate (one half of) the condition for $X \in Obj \underline{C}$ to belong to $\underline{C}^{w \geq 0}$ one should translate the condition $X \perp j_!(Obj \underline{E}^{t \leq 0})$ into certain condition of the type $\underline{E}(F(X), Y) = \{0\}$, $\forall Y \in Obj \underline{E}^{t \leq 0}$ for a certain functor $F : \underline{C} \rightarrow \underline{E}$. The problem is that F should be left adjoint to $j_!$ i.e. it is not a part of our gluing data (cf. also parts 3-4 of Proposition 4.4.5). One would also have a similar (actually, a dual) problem for w that is right adjacent to t .

5. One could also try to 'glue' coniveau spectral sequences (this would correspond to gluing of the corresponding Gersten weight structures as defined in [Bon10]). Yet this could be difficult; see the problem with gluing of adjacent structures described above.

8.3 Multiple compositions of t - and weight truncations

Now suppose that \underline{C} is endowed with some weight structures w_i , $1 \leq i \leq m$, such that there exist left adjacent t -structures t_i .

Then applying parts 7, 8 of Theorem 4.4.2 one can easily and naturally express the functors represented by all possible compositions of t_i -truncations as certain images (as in loc.cit).

For example, applying part 7 of loc.cit. twice we obtain

$$\underline{C}(X, \tau_{1, \leq i}(\tau_{2, \leq j}Y)) \cong \text{Im}(\underline{C}(w_{1, \leq i}(w_{2, \leq j}X), Y) \rightarrow \underline{C}(w_{1, \leq i+1}(w_{2, \leq j+1}X, Y)); \quad (44)$$

for all $i, j \in \mathbb{Z}$; this isomorphism is functorial in both X and Y . We recall that morphisms of objects can be (non-uniquely) extended to morphisms of their weight decompositions; one can also apply Proposition 2.5.1. In these formulae one can also shift t -truncations by $[l]$, $l \in \mathbb{Z}$, and compose truncations from different sides. For example, (16) is essentially a formula of this sort.

Now we somewhat extend these results. Note that a duality (see part 5 of Remark 4.4.3) could be given by $\Phi(X, Y) = \underline{C}(X, F(Y))$, where $F : \underline{D} \rightarrow \underline{C}$ is an exact functor; in this case $A = Ab$. The case of adjacent structures then corresponds to $F = id_{\underline{C}}$.

So, suppose that \underline{C} , \underline{D} , and \underline{E} are triangulated categories, let $F : \underline{D} \rightarrow \underline{C}$, $G : \underline{E} \rightarrow \underline{D}$ be exact functors; let w_1, w_2 be weight structures for \underline{C} , t_1 be a t -structure for \underline{D} and t_2 be a t -structure for \underline{E} , suppose that they satisfy the orthogonality conditions: $\underline{C}(\underline{C}^{w_1 \leq 0}, F(\underline{D}^{t_1 \geq 1})) = \underline{C}(\underline{C}^{w_1 \geq 1}, F(\underline{D}^{t_1 \leq 0})) = \underline{C}(\underline{C}^{w_2 \leq 0}, F \circ G(\underline{E}^{t_2 \geq 1})) = \underline{E}(G \circ F(\underline{C}^{w_2 \geq 1}), \underline{E}^{t_2 \leq 0}) = 0$. Then one can express $\underline{C}(X, F(\tau_{1, \leq i}G(\tau_{2, \leq j}Y)))$ for $Y \in \text{Obj} \underline{E}$ as a certain image similar to (44).

Certainly, one may also consider compositions of more than two exact functors.

8.4 A strong weight complex functor for triangulated categories that admit f -triangulated enhancements

Now we check that the strong weight complex functor t exists if there exists an f -category enhancement of our category (we will define this notion very soon); see part 3 of Remark 3.3.4. The argument below was kindly communicated to the author by prof. A. Beilinson. To make our notation compatible with those of [Bei87] we will denote our basic triangulated category (which is usually \underline{C}) by D . As usual, D is endowed with a weight structure w .

The plan of the construction is the following one. Suppose that there exists an f -category DF over D . In particular, this yields the existence of the 'forgetting of filtration' functor $\omega : DF \rightarrow D$. We describe a class of objects $DF^s \subset \text{Obj} DF$ such that:

- (i) any object of $X \in \text{Obj} D$ can be 'lifted' to an element of $X^* \in DF^s$;
- (ii) For every $M, N \in DF^s$ the map $DF(N, M) \rightarrow D(\omega(N), \omega(M))$ is surjective;
- (iii) There exists a functor $e : DF \rightarrow C^b(D)$ such that $e(DF^s) \subset C^b(\underline{Hw})$ and for any $M, N \in DF^s$ the functor e maps $\text{Ker} DF(N, M) \rightarrow$

$D(\omega(N), \omega(M))$ to morphisms that are homotopic to 0.

We will denote the induced functor $DF \rightarrow K^b(D)$ by e' .

Then $X \rightarrow e(X^*)$ yields an additive functor $T : D \rightarrow K$ where K is a certain triangulated category isomorphic to $K^b(\underline{Hw})$. Indeed, by (ii) any two choices of X' are connected by (possibly, non-unique) morphisms. By (iii) these morphisms become canonical isomorphisms after the application of e' . Hence it suffices to take K being the category obtained from $K^b(\underline{Hw})$ by factorizing by these isomorphisms. Indeed, this family respects coproducts since ω and e do.

Now we recall the relevant definitions of the Appendix of [Bei87].

Definition 8.4.1. I A triangulated category DF will be called a filtered triangulated one if it is endowed with strict triangulated subcategories $DF(\leq 0)$ and $DF(\geq 0)$; an exact autoequivalence $s : DF \rightarrow DF$; and a morphism of functors $\alpha : id_{DF} \rightarrow s$, such that the following axioms hold (for $DF(\leq n) = s^n(DF(\leq 0))$ and $DF(\geq n) = s^n(DF(\geq 0))$).

(i) $DF(\geq 1) \subset DF(\geq 0)$; $DF(\leq 1) \supset DF(\leq 0)$; $\cup_{n \in \mathbb{Z}} DF(\geq n) = \cup_{n \in \mathbb{Z}} DF(\leq n) = DF$.

(ii) For any $X \in Obj DF$ we have $\alpha_X = s(\alpha_{s^{-1}X})$.

(iii) For any $X \in Obj DF(\geq 1)$ and $Y \in Obj DF(\leq 0)$ we have $DF(X, Y) = \{0\}$; whereas α induces an isomorphism $DF(Y, s^{-1}X) \cong DF(sY, X) \rightarrow DF(Y, X)$.

(iv) Any $X \in Obj DF$ can be completed to a distinguished triangle $A \rightarrow X \rightarrow B$ with $A \in Obj DF(\geq 1)$ and $B \in Obj DF(\leq 0)$.

II DF is called an f-category over D if $D \subset DF$; $Obj D = Obj DF(\leq 0) \cap Obj DF(\geq 0)$.

III We will denote by ω (see Proposition A3 of [Bei87]) the only exact functor $DF \rightarrow D$ such that:

(i) Its restrictions are right adjoint to the inclusion $D \rightarrow DF(\leq 0)$ and left adjoint to the inclusion $D \rightarrow DF(\geq 0)$ respectively.

(ii) $\omega(\alpha_X)$ is an isomorphism.

(iii) $DF(X, Y) = D(\omega X, \omega Y)$ for any $X \in Obj DF(\leq 0)$, $Y \in Obj DF(\geq 0)$.

A simple example of this axiomatics is described in Example A2 loc. cit.

By Proposition A3 loc. cit. there also exist exact functors $\sigma_{\geq n} : DF \rightarrow DF(\geq n)$, and $\sigma_{\leq n} : DF \rightarrow DF(\leq n)$ that are respectively right and left adjoint to the corresponding inclusions. We denote $gr_F^{[a,b]} := \sigma_{\leq b} \sigma_{\geq a}$, $gr_F^a = gr_F^{[a,a]}$. Note that there exist canonical and functorial (in X) morphisms $d : \sigma_{\leq 0} X \rightarrow \sigma_{\geq 1} X[1]$ that can be completed to a distinguished triangle in I(iv) of Definition 8.4.1.

Now we define e . For $M \in \text{Obj}DF$ the complex $e(M)$ has components equal to $s^{-a}gr_F^a M[a]$ (this lies in $\text{Obj}D \subset \text{Obj}DF$), the differential will be equal to $s^{-a-1}(s(d') \circ \alpha_{gr_F^a})[a]$; here d' is the boundary map of the canonical triangle $gr_F^{a+1} \rightarrow gr_F^{[a,a+1]} \rightarrow gr_F^a \xrightarrow{d'} gr_F^{a+1}[1]$. M is a complex indeed by the axiom I(ii). We have $e(s(X)) \cong e(X)$.

Now for a weight structure w on D we define $DB^s = e^{-1}(C^b(\underline{Hw}))$ i.e. we demand $gr_F^a(X) \in s^a D^{w=a}$.

We will use the following statement.

Lemma 8.4.2. *For every $M, N \in DF^s$ the map $\alpha_* DF(N, M) \rightarrow DF(N, s(M))$ is surjective; all $DF(N, s^a(M)) \rightarrow DF(N, s^{a+1}(M))$ for $a > 0$ are bijective.*

Proof. Set $P = \text{Cone}(\alpha_M : M \rightarrow s(M))$. By the long exact sequence for $DF(N, -)$, it suffices to show that $DF(N, s^a(P)[b]) = \{0\}$ for $a + b \geq 0$. Since $s^a M[-a] \in DF^s$, it suffices to show that $DF(N, P[b]) = \{0\}$ for $b \geq 0$.

By devissage, we can assume that $gr_F^a M$ and $gr_F^b N$ vanish for $a \neq m$, $b \neq n$, $m, n \in \mathbb{Z}$. In other words, $M = s^m(K)[-m]$, $N = s^n(L)[-n]$ for some $K, L \in \underline{C}^{w=0} \subset \text{Obj}D \subset \text{Obj}DF$.

One has $DF(N, P[b]) = D(L[-n], \omega(\sigma_{\geq n} P)[b])$. To see that this group vanishes, consider 3 cases.

- (a) Suppose that $n > m + 1$. Then $\sigma_{\geq n} P = 0$.
- (b) Suppose $n \leq m$. Then $\sigma_{\geq n} P = P$, so $\omega(\sigma_{\geq n} P) = \omega(P) = 0$.
- (c) Suppose $n = m + 1$. Then $\sigma_{\geq n} P = s(M)$, so $\omega(\sigma_{\geq n} P) = K[-m]$ and $D(N, P[b]) = D(L[-n], K[-m + b]) = D(L, K[b + 1]) = \{0\}$ since w is a weight structure. \square

Now (ii) follows from Lemma 8.4.2 immediately since for any $X, Y \in \text{Obj}DF$ we have $DF(X, s(Y)) \cong D(\omega(X), \omega(s^n Y)) \cong D(\omega(X), \omega(Y))$ for n large enough by parts III(ii), III(iii) of Definition 8.4.1.

(ii) easily yields (i). Indeed, we can prove the statement for $X \in D^{[i,j]}$ by the induction on $j - i$. We have obvious inclusions $D^{w=i} \rightarrow DF^s$ (that split ω).

To make the inductive step it suffices to consider $X \in D^{[0,m]}$ for $m > 0$. Then $X^{w \leq 0}$ and $X^{\geq 1}$ can be lifted to DF^s by the inductive assumption. The map $X^{w \leq 0} \rightarrow X^{\geq 1}$ lifts to DF^s by Lemma 8.4.2; its cone will belong to DF^s and so will be a lift of X .

Now we verify (iii). By Lemma 8.4.2, for $M, N \in DF^s$ we have

$$\text{Ker}(DF(N, M) \rightarrow D(\omega(N), \omega(M))) = \text{Ker}(\alpha_* DF(N, M) \rightarrow DF(N, s(M))).$$

Since $\omega(M) \rightarrow \omega(s(M))$ is an isomorphism, we obtain (iii). Hence T is a well-defined functor.

Now we note that T is an 'enhancement' of our 'weak' weight complex functor t . Indeed, T and t coincide on \underline{Hw} ; both of them respect weight decompositions of objects and morphisms in a compatible way.

Lastly, to check that T is an exact functor one should apply the method of Remark 3.3.2 and of the proof of Theorem 3.3.1. We obtain: it suffices to lift any distinguished triangle $C \rightarrow X \rightarrow X'$ so that the sequence $e(C^*) \rightarrow e(X^*) \rightarrow e(X'^*)$ splits termwisely (in $C^b(\underline{Hw})$). Now, to find such lifting it suffices to choose the weight decompositions of X and X' arbitrarily; choose a weight decomposition of C as in the proof of part I of Theorem 3.3.1; and lift to DB^s the map $t(C) \rightarrow t(X)$ as in the proof of (i). Then the map $a^* : C^* \rightarrow X^*$ will become split surjective after the application of each gr_F^a ; hence we can choose $\text{Cone } a^* \in DF^s$ as a lift for X' . This yields the lift desired.

Hence T is a strong weight complex functor for D, w . This argument is a certain weight structure counterpart of Proposition A5 of [Bei87].

It also seems possible that an f -category enhancement of D would allow to define certain *higher truncation* functors; see Conjecture 6.3.1.

8.5 Possible variations of the weight complex functor; reduction modulo p

Now we try to tell whether the main results of this paper could be generalized to a more general setting. We cannot prove any if and only if conditions; however we try to clarify the picture. Since we include this subsection only to explain our choice of definitions, it is rather sketchy.

First we study the question where do exact conservative functors come from.

Suppose that $f : \underline{C} \rightarrow \underline{C}'$ is an exact functor (here $\underline{C}, \underline{C}'$ are triangulated categories). We denote by $\text{Ker } f$ the class of morphisms that are mapped to 0 by f_* . $\text{Ker } f$ is a (two-sided) ideal of $\text{Mor } \underline{C}$ (see Definition 3.1.1).

It is easily seen that if f is conservative if and only if $\text{id}_X \notin \text{Ker } f$ for any $X \in \text{Obj } \underline{C}$. Note that in this case $\text{Ker } f$ could be called a radical ideal since for any $X \in \text{Obj } \underline{C}$, $s \in \underline{C}(X, X) \cap \text{Ker } f$, $\text{id}_X + s$ will be an automorphism.

Now we study an inverse problem: which ideals can correspond to conservative exact functors. Unfortunately, it seems that there does not exist a nice way to kill morphisms in an arbitrary I unless \underline{C} has a differential graded enhancement. So we suppose that $\underline{C} = \text{Tr}^+(D)$ for a differential graded category D ; an ideal $I \triangleleft \text{Mor } \underline{C}$ comes from a differential graded nilpotent (or formally nilpotent in an appropriate sense) ideal I' of $\text{Pre-Tr}^+ D$. Then one can form a category $\underline{C}' = \text{Tr}^+(D/I')$; using a certain spectral sequence argu-

ment for representable functors X_* for $X \in \text{Obj} \underline{C}$ similar to those described below (for realizations) one can verify that the natural differential graded functor $\underline{C} \rightarrow \underline{C}'$ is conservative. However one cannot hope for a spectral sequence for a realization H unless $H(I)$ belongs to some nice radical ideal (probably more conditions are needed). Note that this is obviously the case for representable functors.

We describe one of the cases when it makes sense to construct such a theory (and which does not come from a weight structure). Let \underline{C}, D be pro- p -categories (i.e. the morphism set is an abelian pro- p -group for any pair of objects), $\underline{C} = \text{Tr}^+ D$, and $I' = p\text{Mor}(\underline{C})$. Let $H = \text{Tr}^+(E)$ for a differential graded functor $E : D \rightarrow B(\text{pro} - p - \text{Ab})$, where $B(\text{pro} - p - \text{Ab})$ is the 'big' category of complexes of abelian profinite p -groups (see subsection 6.4). Then the complex that computes $H(X)$ for $X \in \text{Obj} \underline{C}$ has a natural filtration by subcomplexes given by $p^i E$. These subcomplexes correspond to the functors $\text{Tr}^+(p^i E)$ and the factors of the filtration are quasi-isomorphic to those calculating the functors $F_i = \text{Tr}^+(p^i E / p^{i+1} E)$. It remains to note that F_i can be factorized through the natural functor $\underline{C} \rightarrow \text{Tr}^+(D/p)$. Hence in this case the spectral sequence of a filtered complex has properties similar to those of the spectral sequence S in §6.4; F_i are similar to truncated realizations (see §6.4 above and §7.3 of [Bon09]).

8.6 Three types of filtrations for triangulated categories; the relation of $\mathcal{W}_{\text{Chow}}$ with the weight filtration for motives

The author believes that there exist three equally important types of filtrations for triangulated categories: t -structures, weight structures and *horizontal structures*. Here a (left) horizontal structure denotes a filtration of C by full triangulated subcategories C_i such that for any i the inclusion $C_i \rightarrow C$ admits a (left) adjoint (we call a filtration of this type horizontal since it is shift-invariant). Any filtration of any of the three types of described defines canonical functorial spectral sequences for any cohomology of objects.

It could be interesting to study various 'configurations' of the structures of these types. In particular, we know that DM_{gm}^{eff} and $DM_{gm}^{eff} \mathbb{Q}$ support the corresponding Chow weight structures (see §6.6). Conjecturally, $DM_{gm}^{eff} \mathbb{Q}$ should also support the mixed motivic t -structure (cf. §7.2) and the (horizontal) *weight filtration*. Here the latter is given by C_i that are generated (as triangulated categories) by (the cohomology) $H_{MM}^j(X)$, $X \in \text{Obj} \text{Chow}^{eff} \mathbb{Q}$, $j \geq -i$. Mixed motivic cohomology 'should be' strictly compatible with étale cohomology; for $P \in \text{SmPrVar}$ one should have

$H_{MM}^j(M_{gm}(P)) = 0$ for $j > 0$; hence $C_{-1} = \{0\}$. We obtain that the weight filtration and t_{MM} induce the same filtration on $Chow^{eff}\mathbb{Q}$. Moreover, on $MM = \underline{Ht}_{MM}$ the weight filtration should induce the same filtration as the Chow weight structure. Respectively, one should have: $X \in DM_{gm}^{eff}\mathbb{Q}^{w \leq 0}$ whenever for all $j \in \mathbb{Z}$ we have $H_{MM}^{-j}(X) \in C_j$.

In [Wil09at] the (conjectural) picture described above was justified (in the case when k is a number field) for the triangulated category $DAT \subset DM_{gm}^{eff}\mathbb{Q}$ (of so-called Artin-Tate motives). It was also shown that the restriction of w_{Chow} to DAT can be completely characterized in terms of weights of singular homology. Actually, this corresponds to the fact that the triangulated category DHS of mixed Hodge complexes can be endowed with a weight structure and also with a 'classical weight filtration'; these filtrations and the 'canonical' t -structure for DHS are connected by the same relations as those that 'should' connect the corresponding filtrations of motives. It can be easily seen the mixed Hodge realization respects weight structures mentioned; it should also 'strictly respect' them (and this was essentially proved in [Wil09at] for Artin-Tate motives).

It could also be interesting to study the relation between the homotopy t -structure and the *slice filtration* on DM_-^{eff} ; see [HuK06].

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